

Consistency Strengths of Modified Maximality Principles

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Abstract

The Maximality Principle MP is a scheme which states that if a sentence of the language of ZFC is true in some forcing extension $V^{\mathbb{P}}$, and remains true in any further forcing extension of $V^{\mathbb{P}}$, then it is true in all forcing extensions of V . A modified maximality principle MP_{Γ} arises when considering forcing with a particular class Γ of forcing notions. A parametrized form of such a principle, $\text{MP}_{\Gamma}(X)$, considers formulas taking parameters; to avoid inconsistency such parameters must be restricted to a specific set X which depends on the forcing class Γ being considered. A stronger *necessary* form of such a principle, $\Box\text{MP}_{\Gamma}(X)$, occurs when it continues to be true in all Γ forcing extensions.

This study uses iterated forcing, modal logic, and other techniques to establish consistency strengths for various modified maximality principles restricted to various forcing classes, including CCC , COHEN , COLL (the forcing notions that collapse ordinals to ω), $< \kappa$ directed closed forcing notions, etc., both with and without parameter sets. Necessary forms of these principles are also considered.

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Introduction

The **Maximality Principle** (MP) states that if a sentence of ZFC is true in some forcing extension $V^{\mathbb{P}}$ of V and remains true in any subsequent forcing extension of $V^{\mathbb{P}}$, then it is true in V . An equivalent form of this principle says that the given sentence must then be true in all forcing extensions of V . This principle (or any of its variations) can play a role analogous to that of other forcing axioms such as MA or PFA in deciding questions which are left unanswered by ZFC alone (for example, concerning the size of the continuum). It is a **dense-set free** forcing axiom—it doesn’t require that a filter meet some family of dense sets; rather, it is expressed in terms of the forcing relation itself. The maximality principle is discussed extensively in [HAM1], and some variations are explored in [SV].

The principle MP can be cast in the language of modal logic by regarding models of ZFC as possible worlds and defining one such world to be accessible from another if the first is a forcing extension of the second. (This relation is transitive and reflexive, giving rise to the $S4$ axiom system of modal logic.) This gives interpretations of the modal concepts of possibility and necessity. A statement ϕ is possible ($\Diamond\phi$ in the notation of modal logic) if it is true in some forcing extension and necessary (denoted by $\Box\phi$) if it is true in every forcing extension. A sentence is **possibly necessary**, or **forceably necessary** if it is true in some forcing extension and remains true in any subsequent forcing extension. So MP says that if a sentence is forceably necessary then it is necessary. It is not a formula of ZFC: it is a scheme, the collection of all instances of the statement “ $\Diamond\Box\phi$ implies $\Box\phi$ ” (or its $S4$ equivalent, “ $\Diamond\Box\phi$ implies ϕ ”) where ϕ ranges over all sentences in the language of ZFC.

The Maximality Principle is equiconsistent with ZFC ([HAM1]), but it has two basic variations of greater consistency strength. The first is a variation in which any formula ϕ occurring in an instance of the scheme is allowed to take

parameters. This variation was denoted \mathbf{MP} (“boldface” MP) in [HAM1], in which it was shown that the most general class of parameters which does not render the principle patently false is $H(\omega_1)$, the set of hereditarily countable sets. It is easy to see that such a principle with parameters allowed from a larger set would be false. Indeed, a hereditary uncountable set can be forced to have countable transitive closure in any further extension, so if it were allowed as a parameter in MP, it would already be countable, contradicting the assertion of its uncountability. Since any hereditarily countable set can be coded by a real, we will denote this variation of MP by $\mathbf{MP}(\mathbb{R})$, which expresses the idea that any statement in the language of set theory with arbitrary real parameters that is forceably necessary is necessary. In exploring modifications to the maximality principle the new notation, $\mathbf{MP}(X)$ for MP with parameter set X , is more perspicuous when discussing how different modifications of MP lead to different natural classes of parameters and on the other hand, how changing the class of parameters while keeping the same class of forcing notions leads to a different maximality principle, possibly of different consistency strength.

The third and final basic form, of greatest consistency strength, is the Necessary Maximal Principle, or $\Box\mathbf{MP}(\mathbb{R})$, which says that $\mathbf{MP}(\mathbb{R})$ is necessary — it holds in V and every forcing extension $V^{\mathbb{P}}$. This principle has greater strength (still unknown exactly, but it has been bounded above and below by consistency strengths that put it well up in the hierarchy, along with Woodin cardinals and Projective Determinacy, see [HAM1]). This is because parameters of the formula ϕ are not restricted to those with interpretation in V ; they may appear as a result of the forcing that produces the model in which $\mathbf{MP}(\mathbb{R})$ is interpreted and remains necessarily true.

Modified maximality principles arise if one considers possible worlds (models of ZFC) to be accessible only if they are extensions obtained by forcing with forcing notions of specified classes. If Γ is such a restricted class of forcing notions, such an accessibility relation can be expressed between models M_1 and M_2 by saying that M_2 is a Γ -**forcing extension** of M_1 (that is, that $M_2 = M_1[G]$ where G is M_1 -generic over some \mathbb{P} where $\mathbb{P} \in \Gamma$). A formula ϕ is Γ -**necessary** (denoted by $\Box_{\Gamma}\phi$) when it is true in all Γ -forcing extensions. (A related concept, used in [SV], defines ϕ to be Γ -**persistent** if, whenever ϕ is true, it is then true in all Γ -forcing extensions. So if ϕ is Γ -persistent it may not be true, in which case it need not be Γ -necessary. A formula ϕ is Γ -**forceable** (denoted by $\Diamond_{\Gamma}\phi$) when it is true in some Γ -forcing extension. (Note that, as in other interpretations of possibility and

necessity, that \Diamond and \Box are dual to each other— $\Diamond\phi$ can be defined as $\neg\Box\neg\phi$.) A formula ϕ is Γ -**possibly necessary** or Γ -**forceably necessary** (denoted by $\Diamond_\Gamma\Box_\Gamma\phi$) if it is Γ -forceable that ϕ is Γ -necessary. (The determination as to whether a forcing notion \mathbb{P} in a Γ -forcing extension is itself in Γ is made *de dicto*. That is, Γ is a definable class in ZFC, as such, its members are formally determined by satisfying its defining formula. So this formula is interpreted in the model of ZFC in which \mathbb{P} will be forced with.) The modified maximality principle that arises with these notions is denoted MP_Γ , which says that if a formula of ZFC is true in some Γ -forcing extension $V^\mathbb{P}$, (that is, where $\mathbb{P} \in \Gamma$) of V and remains true in any subsequent Γ -forcing extension of $V^\mathbb{P}$, then it is Γ -necessary in V (hence true, if Γ includes the trivial forcing notion $\{\emptyset\}$, as it almost always will). In terms of the symbols just introduced, the principle MP_Γ can be expressed as the scheme $\Diamond_\Gamma\Box_\Gamma\phi \implies \Box_\Gamma\phi$ where ϕ can be any statement in the language of ZFC.

If a modified maximality principle is allowed to take parameters, then the notation indicated above will be used to indicate a modified maximality principle with parameters from the class specified in parentheses. Thus, a formula is **ccc-necessary** if it is true in all CCC-forcing extensions. The principle $\text{MP}_{\text{ccc}}(H(2^\omega))$ then says that any formula with a parameter from $H(2^\omega)$ —the sets hereditarily of cardinality less than 2^ω —that is CCC-forceably necessary is CCC-necessary. It is a result of [HAM1] that MP_{ccc} with parameters is consistent (relative to the Lévy Scheme) only as long as the parameters are taken from $H(2^\omega)$, so this is a **natural class** of parameters for this particular principle. This means that two conditions are met. First, that any instance of the principle with any parameter in this class is consistent, and second, that any instance of the principle with any parameter not in this class is inconsistent. In what follows, the first order of business upon introducing a modified maximality principle will be to determine the natural classes of parameters for which it is consistent. The class of CCC forcing notions itself will be simply denoted by the abbreviation CCC, and other classes of forcing notions will be similarly abbreviated.

Modifications to the Necessary Maximality Principle will be denoted using the same system. For example $\Box\text{MP}_{\text{ccc}}(H(2^\omega))$ says that $\text{MP}_{\text{ccc}}(H(2^\omega))$ is CCC-necessary (the box operator is implicitly restricted to CCC in this notation). This way of forming new principles can be applied equally to any other restricted type of forcing.

In this work I will try to answer questions concerning the consistency strengths of these modified maximality principles. The work is organized

around the various classes of forcing notions (forcing notions) that occur in various modifications of the maximality principle. After a chapter on general concepts, Chapter 2 is devoted to classes of forcing notions that preserve specified statements in the language of ZFC, chapter 3 to the class of CCC forcing notions, chapter 4 to Cohen forcing, chapter 5 to forcing notions that collapse cardinals to ω , and chapter 6 to forcing notions parametrized by large cardinals. A goal of this work has been to establish the consistency strength of any given maximality principle with the most extensive class of parameters that results in a consistent principle.

Chapter 1

Fundamental notions

1.1 Modal logic of forcing extensions

1.1.1 Set-theoretic semantics

The modal logic of forcing extensions is a notational convenience for proving theorems in ZFC. (A good introduction to the formalism of modal logic used here is [FM].) The notation is called modal logic because the symbols used, \Box and \Diamond , obey certain modal logic axioms when interpreted in the forcing context. In addition, regarding models of ZFC as possible worlds corresponds well with Kripke model semantics. For any formula ϕ , $\Box\phi$ says that ϕ is true in all forcing extensions: for all \mathbb{P} , $V^{\mathbb{P}} \models \phi$. $\Diamond\phi$ says that ϕ is true in some forcing extensions. As stated in the introduction, this notation can be relativized to specific classes of forcing notions. This notation adds defined symbols to the language of ZFC. But saying ϕ is true in a forcing extension is already an abbreviation of a sentence in the language of ZFC that involves an abstract “forcing language” using \mathbb{P} -names based on some partial order \mathbb{P} . So in adding modal symbols, we are just continuing a tradition of adding another degree of separation from the primitive language $\{\in\}$. In order to benefit from the ability to think intuitively, using and mining familiar concepts founded in Kripke models, we may regard forcing extensions as accessible possible worlds. But while harvesting intuitions provided by the modal symbols’ interpretations as necessity and possibility, one should remain centered in the realization that these are just statements in the language of ZFC being interpreted in a single model, replete with names for sets in all forcing extensions.

For us, it is primarily the economy of expression of maximality principles and the arguments they require that leads to the use of these modal symbols. Recall that their definitions given in the introduction are in fact set-theoretic. Thus \Box , \Diamond and their relativized counterparts are defined symbols added to the language of ZFC; they can be safely eliminated from any formal argument in which they are used.

An equivalent definition of \Box and \Diamond can be made in terms of complete Boolean algebras, a commonly used foundation for forcing arguments. For a complete discussion of this approach see [BELL]. If one is working in a Boolean-valued model $V^{\mathcal{B}}$ of ZFC, with Boolean values in the Boolean algebra \mathcal{B} , the Boolean value of a formula ϕ is denoted $\llbracket \phi \rrbracket^{\mathcal{B}}$. If Γ is a class of complete Boolean algebras, we have dual definitions: $\Diamond_{\Gamma}\phi$ if and only if there exists some \mathcal{B} in Γ such that $\llbracket \phi \rrbracket^{\mathcal{B}} \neq 0$, and $\Box_{\Gamma}\phi$ if and only if for all \mathcal{B} in Γ , $\llbracket \phi \rrbracket^{\mathcal{B}} = 1$. All results herein will follow from this definition, including results on iterated forcing, by translating forcing notions to the corresponding regular open algebras. In fact, a Boolean-valued model approach to forcing has a distinct advantage. It allows one to formalize the concept of “forcing over V ” by establishing a relationship between V and $V^{\mathcal{B}}$. Since this occurs within ZFC, it applies over any model. This frees one from having to base a definition of forcing on countable transitive models.

1.1.2 S4 forcing classes

Given this set-theoretic definition of the modal operators, we can ask if the operators in fact behave according to various axioms of modal logic. In our interpretation, an axiom of modal logic will be a scheme in ZFC holding true for any formula of ZFC when substituted into the scheme. If these schemes are true in ZFC, we can conveniently use them to prove assertions regarding maximality principles. The relevant axioms for our purposes will be the **K**, **S4** and **S5** axiom schemes. The **K** axiom schemes are

- (1) $\Diamond_{\Gamma}\phi \leftrightarrow \neg\Box_{\Gamma}\neg\phi$
- (2) $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$;

the **S4** axiom schemes include the **K** axiom schemes as well as

- (3) $\Box\phi \rightarrow \phi$ and
- (4) $\Box\phi \rightarrow \Box\Box\phi$;

and the **S5** axiom schemes include

$$(5) \Diamond\phi \rightarrow \Box\Diamond\phi$$

together with (1)–(4).

Remark. By the duality between \Box and \Diamond , this last axiom is equivalent to (5') $\Diamond\Box\phi \rightarrow \Box\phi$, which is the maximality principle MP if these operators are given the suggested set theoretic meaning.

We now interpret these schemes in the context of the forcing relation relativized to a fixed class Γ of forcing notions. If these schemes are interpreted as schemes in ZFC, then the S4 axiom schemes are intended to hold for all formulas of the language of ZFC, with arbitrary parameters allowed. On the other hand, the S5 schemes are only consistent in ZFC when the parameters for the formulas are restricted to specific sets, depending on the specific class of forcing notions used in defining the modal operators. This is due to the equivalence of the S5 scheme with the modified maximality principle associated with the forcing class. This point is important, as the maximality principles we work with will invariably impose restrictions on the parameters that can be used in order to avoid inconsistencies.

We first show that modal operators defined from any such class of forcing notions obeys the K scheme. From their definitions, they clearly obey the duality of the classical modal operators. Specifically,

Lemma 1.1. *If Γ is any class of forcing notions and ϕ is any formula in the language of ZFC, then $\Diamond_\Gamma\phi \leftrightarrow \neg\Box_\Gamma\neg\phi$.*

Proof. If ϕ holds in some Γ -forcing extension $V[G]$, then, for its negation to be Γ -necessary, $\neg\phi$ would have to hold in $V[G]$ as well, a contradiction. And, conversely, if $\neg\phi$ is Γ -necessary there can be no extension $V[G]$ in which ϕ is true. \square

Lemma 1.2. *Let Γ be any class of forcing notions. If ϕ and ψ are any formulas in the language of ZFC, then $\Box_\Gamma(\phi \rightarrow \psi) \implies (\Box_\Gamma\phi \rightarrow \Box_\Gamma\psi)$.*

Proof. Suppose it is false that $\Box_\Gamma\phi \rightarrow \Box_\Gamma\psi$, i.e., $\Box_\Gamma\phi$ is true but $\Box_\Gamma\psi$ is false. Then there is a Γ -forcing extension $V[G]$ where ϕ is true and ψ is false. Then $V[G] \models \neg(\phi \rightarrow \psi)$. So $\Box_\Gamma(\phi \rightarrow \psi)$ must be false. \square

The class Γ is **closed under two-step iterations** if, whenever \mathbb{P} is in Γ and $V^\mathbb{P} \models \text{“}\dot{\mathbb{Q}} \text{ is in } \Gamma\text{”}$ ¹, then $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ . We will say Γ is **adequate**

¹As noted earlier, deciding whether a forcing notion is in a defined class will always be made *de dicto* in the forcing extension in which the forcing notion is to be forced with.

if Γ is Γ -necessarily closed under 2-step iterations and Γ -necessarily contains trivial forcing, that is, the forcing notion $\{\emptyset\}$ consisting of only one condition. Examples of such classes include the class of all forcing notions, CCC (the class of all forcing notions which satisfy the countable chain condition), PROPER (the class of all proper forcing notions) and CARD (the class of all forcing notions that preserve all cardinals).

Remark. By the definition of adequacy, if Γ is adequate it must then be Γ -necessarily so (i.e., this continues to hold in any Γ -forcing extension). Again, these axioms are schemes of ZFC, allowing arbitrary parameters for the formula in any instance.

Lemma 1.3. *The forcing classes CCC, PROPER, and CARD are all adequate.*

Proof. They clearly all contain trivial forcing. The classes CCC and PROPER are well known to be necessarily closed under two-step iterations. And, the two-step iteration of cardinal-preserving forcing notions will itself preserve cardinals, hence it will be in CARD. \square

Recalling our definitions of Γ -necessary and Γ -forceable, define Γ to be a **K or S4 forcing class** if the operators \Box_Γ and \Diamond_Γ satisfy the K or S4 axioms of modal logic, respectively, and if this is Γ -necessary (i.e., this continues to hold in any Γ -forcing extension).

Theorem 1.4. *Every adequate class of forcing notions is an S4 forcing class.*

Proof. Let Γ be adequate. We first show that the S4 axioms are true, then that they are Γ -necessarily true.

$\Box_\Gamma \phi \rightarrow \phi$: Assume $\Box_\Gamma \phi$, that ϕ is true in every Γ -forcing extension. But Γ includes trivial forcing, whose extension is the same as the ground model, where ϕ is therefore true.

$\Box_\Gamma \phi \rightarrow \Box_\Gamma \Box_\Gamma \phi$: Again assuming $\Box_\Gamma \phi$, we now use the fact that Γ is Γ -necessarily closed under two-step iterations. Let \mathbb{P} be any forcing notion in Γ . Since $\Box_\Gamma \phi$, $V^\mathbb{P} \models \phi$. Now let $\dot{\mathbb{Q}}$ be any \mathbb{P} -name of a forcing notion in $V^\mathbb{P}$ such that $V^\mathbb{P} \models \text{“}\dot{\mathbb{Q}} \text{ is in } \Gamma\text{”}$. Since Γ is closed under two-step iteration, $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ . And since $\Box_\Gamma \phi$, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models \phi$. But $\dot{\mathbb{Q}}$ was arbitrary in $\Gamma^{V^\mathbb{P}}$, so

Γ may be definable in a way that depends on the model; for example, a forcing notion that is not CCC in the ground model may become CCC in a forcing extension. Consider the forcing notion \mathbb{P} that collapses ω_1 to ω , whose conditions are finite partial injective functions $\omega_1 \rightarrow \omega$. It has antichains of size ω_1 but $V^\mathbb{P} \models \text{“}\mathbb{P} \text{ is CCC”}$.

$V^{\mathbb{P}} \models \Box_{\Gamma} \phi$. And since \mathbb{P} was any forcing notion in Γ , we have $\Box_{\Gamma} \Box_{\Gamma} \phi$. So Γ satisfies the S4 axioms.

Further, since Γ is Γ -necessarily closed under 2-step iterations and trivial forcing, the foregoing argument shows that, in any Γ -extension, Γ satisfies the S4 axioms there as well. So Γ is an S4 forcing class. \square

Remark. Theorem 1.4 cannot be strengthened to “if and only if” for arbitrary models. A counterexample is letting Γ be the class of all forcing, with trivial forcing $\{\emptyset\}$ replaced by some other forcing notion with the same effect, such as $\{1\}$. Then Γ is S4 but not adequate. However, we have the following theorem. We say two forcing notions are **forcing-equivalent** if they both produce the same forcing extensions. And we define the operation of taking all conditions below a given condition in a forcing notion to be **restricting the forcing notion to a condition**.

Theorem 1.5. *If $V = L[A]$ for some set A , then every S4 class closed under forcing-equivalence and the operation of restriction to a condition is adequate.*

Proof. Let Γ be an S4 class of forcing notions. Let ϕ be the first-order statement “ V is a Γ -forcing extension of $L[A]$.” (This statement takes the set A as parameter.) Clearly this is true in any Γ -forcing extension of $L[A]$. So if $V = L[A]$, then $\Box_{\Gamma} \phi$ is true. So by the S4 axiom scheme $\Box_{\Gamma} \phi \longrightarrow \phi$, ϕ is true in $L[A]$. But if $L[A]$ is a Γ -forcing extension of itself, (that is, there is \mathbb{P} in Γ which is trivial below some condition) then trivial forcing is forcing-equivalent to a forcing notion in Γ , hence in Γ . And if, in $L[A]$, \mathbb{P} is any forcing notion in Γ , and \mathbb{Q} is any forcing notion in Γ according to $L[A]^{\mathbb{P}}$, then, by the S4 axiom scheme $\Box_{\Gamma} \phi \longrightarrow \Box_{\Gamma} \Box_{\Gamma} \phi$, ϕ is true in $L[A]^{\mathbb{P} * \dot{\mathbb{Q}}}$. That is, $L[A]^{\mathbb{P} * \dot{\mathbb{Q}}}$ is a Γ -forcing extension of $L[A]$. So $\mathbb{P} * \dot{\mathbb{Q}}$ is forcing equivalent to a forcing notion in Γ , hence in Γ . This, together with trivial forcing being in Γ , can also be shown to be Γ -necessary. For this, notice that Γ is Γ -necessarily S4, and that “ $V = L[A]$ for some set A ” is Γ -necessary. Then repeat the above argument in any Γ -extension. This proves that Γ is an adequate forcing class. \square

The obstacle for a general proof of the equivalence of S4-ness with adequacy seems to be the need to refer to the ground model in the statement ϕ . This can be done in the case where $V = L[A]$ by using A as a parameter in ϕ .

If a class Γ of forcing notions is S4 then the principle MP_{Γ} has two equivalent formulations.

Lemma 1.6. *If Γ is an $S4$ class then the following schemes are equivalent:*

- (1) *For any formula ϕ , $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \Box_\Gamma \phi$*
- (2) *For any formula ϕ , $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \phi$.*

Proof. (1) implies (2): By S4, $\Box_\Gamma \phi \rightarrow \phi$. Combining this with $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \Box_\Gamma \phi$ gives $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \phi$.

(2) implies (1): Substitute $\Box_\Gamma \phi$ for ϕ in $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \phi$ to obtain

$$\Diamond_\Gamma \Box_\Gamma \Box_\Gamma \phi \rightarrow \Box_\Gamma \phi \quad (1.1)$$

By the two S4 axioms, $\Box_\Gamma \Box_\Gamma \phi \leftrightarrow \Box_\Gamma \phi$, so the left side of this implication changes to give $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \Box_\Gamma \phi$. \square

Most of the forcing classes we will work with will be S4, so $\Diamond_\Gamma \Box_\Gamma \phi \rightarrow \phi$ is the form of MP_Γ that will be used most frequently.

Lemma 1.7. *Let Γ be $S4$.*

- (1) *$\phi(x)$ is not Γ -forceably necessary if and only if this fact is Γ -necessary.*
- (2) *If members of Γ are necessarily in Γ , then $\phi(x)$ is Γ -forceably necessary if and only if this fact is Γ -necessary.*

Proof. The right-to-left implications of this lemma are obvious from the S4 axiom $\Box_\Gamma \phi \rightarrow \phi$. So we proceed to proving each of the left-to-right implications.

(1): If there is no Γ -forcing extension $V[G]$ where $\phi(x)$ is Γ -necessary, then in all forcing extensions it is not Γ -forceably necessary. So it is Γ -necessarily not Γ -forceably necessary. Formally,

$$\begin{aligned} \neg \Diamond_\Gamma \Box_\Gamma \phi(x) &\implies \neg \Diamond_\Gamma \Diamond_\Gamma \Box_\Gamma \phi(x) \\ &\implies \Box_\Gamma \neg \Diamond_\Gamma \Box_\Gamma \phi(x). \end{aligned}$$

This last modal derivation makes use of the property of S4 modal logic that replacement of a subformula by an equivalent subformula in a formula leads to an equivalent formula.

(2): We use the additional hypothesis that members of Γ are necessarily in Γ . In this case, two forcing notions \mathbb{P} and \mathbb{Q} of Γ can be iterated in Γ since each remains in Γ under forcing by the other. Such iteration is simply product

forcing. If $\phi(x)$ is Γ -forceably necessary, then there is a Γ -forcing extension $V^{\mathbb{P}}$ where $\phi(x)$ is Γ -necessary. Let \mathbb{Q} be in Γ . Then in $V^{\mathbb{Q}}$, \mathbb{P} still forces $\phi(x)$ to be Γ -necessary (in $V^{\mathbb{Q} \times \mathbb{P}}$, since $\mathbb{P} \times \mathbb{Q} = \mathbb{Q} \times \mathbb{P}$, and $\phi(x)$ is Γ -necessary in $V^{\mathbb{P} \times \mathbb{Q}}$, a Γ -forcing extension of $V^{\mathbb{P}}$. \square

1.1.3 Possibilist quantification and rigid identifiers

Various quantified modal logics exist, to handle whether or not the existence of an object can transcend the confines of the world in which some referring expression is being interpreted. Quantification is **possibilist** if the scope of the quantifier extends beyond the world in which interpretation is occurring. The alternative, **actualist** quantification, is that which limits existence of an object to the world in which any reference to it occurs, so quantifiers have their scope limited to one world. Related to possibilist quantification is the concept of a **rigid** designator, a name or referring expression in the language of a theory whose meaning doesn't change from world to world. This also occurs only when existence of an object is allowed to extend to other possible worlds. Since we are working in ZFC, we will appropriate this terminology to use when talking about forcing extensions simply as models of set theory, not as parts of a Kripke model, just as we have appropriated the modal symbols for necessity and possibility without actually talking about possible worlds or accessibility. Rigidity can be formalized using the technique of predicate abstraction. We will not do so here.

Whenever a model of ZFC has another model of ZFC as an end extension (no new elements are added to any set), it is possible to refer to elements in the first model as though they were in the second. One opts for rigid designation in this discourse, in the sense that the models are different but they have corresponding objects that are structurally the same. This occurs in the set theory literature, for example, when one writes of making a supercompact cardinal, κ , indestructible by Laver's forcing preparation. The name κ is allowed to refer to a cardinal which exists in two different models of ZFC, but because the ground model embeds into the extension, there is no harm in regarding its elements as belonging to the extension. With forcing extensions, new sets can be added that are not in the ground model, but old ones cannot be removed. This property is known as **monotonicity**. In this situation, using possibilist quantification, the Converse Barcan formula of quantified modal logic is valid, which is expressed as either of the dual schemes

- (1) $\Box(\forall x)\phi(x) \rightarrow (\forall x)\Box\phi(x)$ or
- (2) $(\exists x)\Diamond\phi(x) \rightarrow \Diamond(\exists x)\phi(x)$.

We can easily verify the validity of the first scheme when translated into pure ZFC. It simply says that if it is necessary (true in all forcing extensions) that, for all x , $\phi(x)$ is true, then, for all x , $\phi(x)$ is true in all forcing extensions. And this is clear: the ground model contains no element that isn't in all forcing extensions, so the truth of $\phi(x)$, for all x in the ground model, must hold in all forcing extensions.

The concept which is converse to monotonicity is **anti-monotonicity**. It occurs in Γ -forcing extensions when sets not in the ground model are never created. The Barcan formula applies to such forcing classes:

- (1) $(\forall x)\Box\phi(x) \rightarrow \Box(\forall x)\phi(x)$ or
- (2) $\Diamond(\exists x)\phi(x) \rightarrow (\exists x)\Diamond\phi(x)$.

This will generally only occur in a class of forcing notions that only contains trivial forcing. However, bounded quantification over sets will produce an effect analogous to anti-monotonicity, due to absoluteness of the set membership relation.

Lemma 1.8 (Restricted Barcan Formula). *Let Γ be a class of forcing notions, and let A be a set. If for all x in A , $\phi(x)$ is Γ -necessary, then it is Γ -necessary that for all x in A , $\phi(x)$. Formally, $(\forall x \in A)\Box_\Gamma\phi(x) \longrightarrow \Box_\Gamma(\forall x \in A)\phi(x)$.*

Proof. Suppose that for all x in A $\phi(x)$ is Γ -necessary. We want to show that $\Box_\Gamma(\forall x \in A)\phi(x)$, which can be rewritten as $\Box_\Gamma(\forall x)(x \in A \longrightarrow \phi(x))$. So let \mathbb{P} be in Γ , with G a V -generic filter over \mathbb{P} , and let x be in $V[G]$, with $V[G] \models x \in A$. Then $V \models x \in A$ by absoluteness. Then $\phi(x)$ is Γ -necessary by hypothesis. So $V[G] \models \phi(x)$. Thus $V[G] \models (x \in A \longrightarrow \phi(x))$. Since \mathbb{P} was arbitrary in Γ , and x was arbitrary in $V[G]$, this gives $\Box_\Gamma(\forall x)(x \in A \longrightarrow \phi(x))$. \square

1.1.4 Modal logic of forcing Kripke models

This section is added as a bridge to the traditional interpretation of the modal operators, since there is an intuitive interpretation of forcing that

allows one to think of forcing extensions as separate models of ZFC completely outside the universe of the ground model. Although these are not the formal semantics we are using, they can be intuitively thought of as a traditional **Kripke model**, a nonempty set of objects called **possible worlds** together with a binary relation on those objects called an **accessibility relation**. All the possible worlds in a Kripke model interpret the same fixed first-order language, which is augmented with the symbols \Box and \Diamond . For any formula ϕ , $\Box\phi$ asserts that ϕ is **necessary**, or true in every accessible world, while $\Diamond\phi$ asserts that ϕ is **possible**, or true in some accessible world.

Any Kripke model requires a bare-bones modal logic (called **K**) that includes at least all axioms of classical first-order logic (including the rule of inference *modus ponens*), together with the axiom scheme, for all formulas ϕ and ψ ,

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

and the *necessitation* rule of inference

$$\frac{\phi}{\Box\phi}$$

(The necessitation rule means that if the formula ϕ occurs in a proof tableau or proof sequence, then the formula $\Box\phi$ is a valid deduction, since ϕ has been proved according to axioms valid in all worlds, hence ϕ is true in all accessible worlds. In short, if ϕ is provable, then $\Box\phi$. It does *not* say $\phi \rightarrow \Box\phi$.) Additional axioms and rules of inference depend on the type of Kripke model involved. If the accessibility relation of a Kripke model obeys reflexivity and transitivity, it can be shown to obey the **S4** axiom schemes mentioned earlier. If that relation is also symmetric, it obeys the **S5** axiom scheme also given earlier. In fact the accessibility relation we will consider, that of being a Γ -forcing extension, is not symmetric unless Γ consists only of trivial forcing.

We now interpret these ideas in the context of the forcing relation. Given any model M of set theory, there are many models of set theory derivable from M through the mechanism of forcing. We call the Kripke model that is the totality of these models, together with the accessibility relation “ M' is a forcing extension of M ” to be a **forcing Kripke model**. If Γ is a class of forcing notions we can relativize this idea to a Γ -forcing Kripke model as well.

Again, in this work, the interpretation of modal logic symbols and terminology will be within ZFC, hence within the single universe V that models it,

not in this forcing Kripke model of possible worlds. But the correspondence of ideas will be utilized freely. For example, without adding it to ZFC as a new rule of inference, the necessitation rule is used in the form that says that if ϕ is provable it is then true in all models of ZFC.

1.2 Absolutely definable parameter sets

Consider parameters to formulas in instances of MP_Γ . Notice that if a set parameter has a Γ -absolute definition it needn't really be a parameter at all, since it can be eliminated by replacing its occurrence by its definition. So, for any set X whose elements are definable, the principle MP_Γ implies $\text{MP}_\Gamma(X)$ for such X . What is less clear, if X itself is definable but its elements are not required to be so, is whether MP_Γ still implies $\text{MP}_\Gamma(X)$. Situations of this type raise questions such as the following.

Question 1. If $\Gamma \subseteq \text{CARD}$, does MP_Γ imply $\text{MP}_\Gamma(\omega_1)$?

Question 2. If Γ adds no reals, does MP_Γ imply $\text{MP}_\Gamma(\mathbb{R})$?

Note that these questions are about parameters that are contained in ω_1 or \mathbb{R} , not the sets themselves. Thus the principle $\text{MP}_\Gamma(\omega_1)$ asserts that $\Diamond_\Gamma \Box_\Gamma \phi(\alpha)$ implies $\Box_\Gamma \phi(\alpha)$ for any formula ϕ with one free variable, whenever α is in ω_1 .

Formally, we will call a **definition** of a set A a formula $\psi(x)$ such that for all x , $\psi(x)$ if and only if $x = A$. Let Γ be a class of forcing notions. A formula $\phi(x)$ is **Γ -absolute** if for any \mathbb{P} in Γ , and for any x in V , $V^\mathbb{P} \models \phi(x)$ if and only if $\phi(x)$. A set defined by such a formula $\phi(x)$ in all Γ -forcing extensions as well as in V is said to be **Γ -absolutely definable**.

Throughout this chapter Γ will be an S4 forcing class. Let S be a Γ -absolutely definable set (so the definition of S , when interpreted in any Γ -forcing extension of V , results in the same set—no new elements are added). We give some examples to illustrate the idea:

- (1) ω_1^L is absolutely definable over all forcing notions.
- (2) Let Γ_{ω_1} be the class of forcing notions that preserve ω_1 . Then the cardinal ω_1 is Γ_{ω_1} -absolutely definable.

- (3) The set of reals, \mathbb{R} , is not absolutely definable over the class of all forcing notions.

Let $\text{MP}_\Gamma(S)$ be the form of the principle MP_Γ in which statements take parameters from the set S . The fact proven in this chapter is that $\text{MP}_\Gamma \implies \text{MP}_\Gamma(S)$. The strategy in this proof is, for every formula $\phi(\alpha)$ which takes a parameter α in S , to consider the statement $\forall \alpha \in S (\Diamond \Box \phi(\alpha) \longrightarrow \phi(\alpha))$. Since this formula has no free variables (after replacing S with its definition), it is subject to the principle MP_Γ if it holds. If we can show that this statement is Γ -forceably necessary, then, by MP_Γ , it will be true. But taking this over all formulas ϕ we have the scheme $\text{MP}_\Gamma(S)$.

We use a lemma whose proof is simply an exercise in modal logic, using the S4 axioms which characterize the Kripke model of any class of forcing extensions in which accessibility satisfies reflexivity and transitivity. Since Γ , the forcing notion class under discussion, satisfies these, we will let the modal operators \Box and \Diamond be shorthand for \Box_Γ and \Diamond_Γ respectively throughout the rest of this chapter.

Lemma 1.9. *Let Γ be an S_4 forcing class. If $\phi(z)$ is a formula in the language of ZFC with parameter z then the statement “ $\Diamond \Box \phi(z)$ implies $\phi(z)$ ” is forceably necessary.*

Proof. Case I: $\phi(z)$ is forceably necessary ($\Diamond \Box \phi(z)$): Then there is a forcing extension $V[G]$ after which all forcing extensions satisfy $\phi(z)$. Therefore they satisfy that $\phi(z)$ is forceably necessary implies $\phi(z)$.

(Formally, in S4,

$$\Diamond \Box \phi(z) \implies \Diamond \Box (\Diamond \Box \phi(z) \rightarrow \phi(z)).$$

(This derivation is an instance of the S4 tautology

$$\Diamond \Box A \longrightarrow \Diamond \Box (B \longrightarrow A).)$$

Case II: $\phi(z)$ is not forceably necessary ($\neg \Diamond \Box \phi(z)$): Then in all forcing extensions $\phi(z)$ cannot be forceably necessary, whence it is forceably necessary that $\phi(z)$ is not forceably necessary. But in any world where the forceable necessity of $\phi(z)$ is false, the forceable necessity of $\phi(z)$ implies everything,

including $\phi(z)$. (Formally, in S_4 ,

$$\begin{aligned}
 \neg\Diamond\Box\phi(z) &\implies \Box\neg\Box\phi(z) \\
 &\implies \Box\Box\neg\Box\phi(z) \\
 &\implies \Box(\neg\Diamond\Box\phi(z)) \\
 &\implies \Diamond\Box(\neg\Diamond\Box\phi(z)) \\
 &\implies \Diamond\Box(\Diamond\Box\phi(z) \rightarrow \phi(z)).)
 \end{aligned}$$

□

We will say that a class Γ is closed under iterations of order type γ with **appropriate support** if there is an ideal on γ (or other definable choice of support at limit stages) which when used as support for any such iteration produces a forcing notion which is still in Γ . Moreover, we require that if \mathbb{P}_Γ is a γ -iteration of forcing in Γ and $\alpha < \gamma$, then $\mathbb{P}_\gamma = \mathbb{P}_\alpha * \mathbb{P}_{\{\alpha, \gamma\}}$ such that $\mathbb{P}_{\{\alpha, \gamma\}}$ is an iteration of forcing notions of Γ with support in the ideal, as defined in $V^{\mathbb{P}_\alpha}$. An example of such support would be where either direct or inverse limits are always taken; this is Lemma 21.8 in [JECH].

Lemma 1.10. *Let Γ be an S_4 forcing class, Γ -necessarily closed under iterations of arbitrary length with appropriate support. Let S be any set. Let $\phi(x)$ be a formula in the language of set theory with one free variable. Let σ be the formula “ $\forall s \in S (\Diamond\Box\phi(s) \text{ implies } \phi(s))$ ”. Then σ is Γ -forceably necessary.*

Proof. Let ordinal γ be the order type of a fixed wellordering of $S = \langle s_\alpha \mid \alpha < \gamma \rangle$. Let $\sigma(s)$ stand for the formula “ $\Diamond\Box\phi(s)$ implies $\phi(s)$ ”, where s is a parameter of ϕ . By the previous lemma, $\sigma(s_0)$ is forceably necessary, and, more to the point, this fact is a theorem of ZFC, hence, by necessitation, absolute to all Γ -forcing extensions. We begin a finite support γ -iteration of Γ -forcing notions by taking the extension $V^{\mathbb{P}_1}$ in which $\sigma(s_0)$ is necessary. Since $\sigma(s_\alpha)$ is still forceably necessary in $V^{\mathbb{P}_1}$ for any α in γ , we can repeat this construction for another stage, but this time using 1 for α . This gives a model, $V^{\mathbb{P}_2}$, in which both $\sigma(s_0)$ and $\sigma(s_1)$ are necessary, whence $\sigma(s_0) \wedge \sigma(s_1)$ is true and remains true in any extension. Continue in this manner for γ stages: at stage α , let \mathbb{Q}_α be that Γ forcing notion which forces the Γ -necessity of $\sigma(s_\alpha)$. Taking a γ -iteration with appropriate support $\mathbb{P} = \mathbb{P}_\gamma$, we obtain a Γ -forcing extension $V^{\mathbb{P}_\gamma}$. Let G be \mathbb{P} -generic over V . We claim first that, for every α in γ , $V[G] \models \sigma(s_\alpha)$. To see this, choose any α in γ . At stage α , we can factor $\mathbb{P} = \mathbb{P}_\alpha * \mathbb{P}_{TAIL}$, and take G_α as V -generic over \mathbb{P}_α . But at stage α , $V[G_\alpha] \models “\mathbb{Q}_\alpha \text{ forces that } \sigma(s_\alpha) \text{ is } \Gamma\text{-necessary}”$. So refactoring,

$V[G_{\alpha+1}] \models \text{“}\sigma(s_\alpha) \text{ is } \Gamma\text{-necessary”}$. But $V[G]$ is a Γ -forcing extension over $V[G_{\alpha+1}]$ (since Γ is Γ -necessarily closed), so $V[G] \models \text{“}\sigma(s_\alpha) \text{ is } \Gamma\text{-necessary”}$. Formally, $V[G] \models (\forall \alpha \in \gamma) \Box \sigma(s_\alpha)$. By Lemma 1.8, $(\forall \alpha \in \gamma) \Box \sigma(s_\alpha) \rightarrow \Box (\forall \alpha \in \gamma) \sigma(s_\alpha)$. Modus ponens then gives the statement $\Box (\forall \alpha \in \gamma) \sigma(s_\alpha)$, that is, σ is Γ -forceably necessary. \square

Theorem 1.11. *Let Γ be an S_4 forcing class, Γ -necessarily closed under iterations of arbitrary order type with appropriate support. Let S be a set which is Γ -absolutely definable. Then MP_Γ if and only if $\text{MP}_\Gamma(S)$.*

Proof. For any formula ϕ in the language of ZFC, and for any s in S , the formula

$$\forall s \in S (\Diamond \Box \phi(s) \longrightarrow \phi(s))$$

is Γ -forceably necessary by the last lemma. Let ψ be the Γ -absolute definition of S . Then the above formula is equivalent to the formula

$$\forall s (\psi(s) \longrightarrow (\Diamond \Box \phi(s) \longrightarrow \phi(s)))$$

which is free of parameters. So by MP_Γ it is true. \square

Another way to express this result is to say that MP_Γ alone allows free use of parameters in S . And that is the point of this section—that as long as the parameter set is absolutely definable, the actual parameters need not be definable themselves. There is a way to generalize still further. If $\text{MP}_\Gamma(X)$ is assumed a priori, the argument above allows the adjoining to X of a new set of parameters P which is Γ -absolutely definable.

Corollary 1.12. *Let Γ be a class of forcing notions obeying the S_4 axioms of modal logic and closed under arbitrary iterations with appropriate support, and let S be a set Γ -absolutely definable from parameters in X . Then $\text{MP}_\Gamma(X)$ if and only if $\text{MP}_\Gamma(S \cup X)$.*

Proof. In the presence of $\text{MP}_\Gamma(X)$, allow the formula ϕ to take parameters from $X \cup S$. Suppose $\phi(\alpha)$ is forceably necessary. If α is in X , then $\phi(\alpha)$ is true by virtue of $\text{MP}_\Gamma(X)$. Otherwise, α is in S , which is a set whose definition is Γ -absolutely definable from parameters in X . So the sentence $\forall \alpha \in S \cup X (\Diamond \Box \phi(\alpha) \longrightarrow \phi(\alpha))$ is forceably necessary by the above argument. But this sentence can be modified to take only parameters from X by using the definition of S , in which form it will be true by applying $\text{MP}_\Gamma(X)$. The collection of all such sentences then holds, and this collection constitutes the scheme $\text{MP}_\Gamma(S \cup X)$. \square

Finally we consider the union of all Γ -absolutely definable parameter sets which consists of the least initial segment which includes all Γ -absolutely definable sets. We can call this the Γ -**absolutely definable cut** of the universe, V_Γ . This is analogous to the definable cut of the universe, the least initial segment which includes all definable sets. (In contrast with the latter class, however, V_Γ might not be a model of ZFC.)

Corollary 1.13. MP_Γ if and only if $\text{MP}_\Gamma(V_\Gamma)$.

Proof. Assume MP_Γ . Any instance of the scheme $\text{MP}_\Gamma(V_\Gamma)$ will be an instance of the scheme $\text{MP}_\Gamma(S)$ for some Γ -absolutely definable parameter set S , which is true by Theorem 1.11 \square

Again, one can say that MP_Γ alone allows free use of parameters in V_Γ . Returning to the questions asked at the beginning of this section, we can now provide answers. Responding to question 1, let **CARD** be the class of cardinal-preserving notions of forcing.

Corollary 1.14. *If the ordinal α is **CARD**-absolutely definable, then MP_{CARD} implies $\text{MP}_{\text{CARD}}(\omega_\alpha)$. In particular one can freely use $\omega_1, \omega_3, \omega_{\omega_{19}}$, and beyond, as parameter set X in the MP_{CARD} scheme.*

Proof. By Theorem 1.11. \square

Responding to question 2:

Corollary 1.15. *Let Γ be the class of all forcing notions that add no new reals. If MP_Γ holds, then $\text{MP}_\Gamma(\mathbb{R})$ holds.*

Proof. Clearly Γ is adequate, hence S4. Also, \mathbb{R} is Γ -absolutely definable. So apply Theorem 1.11. \square

1.3 Elementary submodels of V

Several theorems in this work require construction of a forcing iteration where each successor stage forces a particular instance of the maximality principle whose model is being sought; the principle is then true in the iterated extension because each instance has been handled at some stage. But in order to define a forcing notion that forces even one instance of a maximality principle requires expressing that a sentence is forceably necessary. The relation

$p \Vdash \phi$ is defined in V with a separate definition for each ϕ , constructed by induction on the complexity of ϕ . So the general relation $p \Vdash \phi$ cannot be expressed as a formula with arguments for p and ϕ . To express this relation requires the scheme consisting of each particular case. This changes for a set model M within V . In this case one can express truth (and forcing) of ϕ in M as a predicate interpreted in V .

Since the forcing predicate is only definable over set models of ZFC, we employ the strategy of [HAM1] of using an initial segment of the universe as an elementary submodel of it. One should be warned that the models of ZFC in which this occurs might possibly be nonstandard (nonwellfounded) models². In particular, we generalize Lemma 2.5 of [HAM1]. We first add a constant symbol δ to the language of ZFC, intended to stand for some ordinal. The following arguments will take place in this expanded language, together with an expanded model of ZFC to interpret this constant.

Let “ $V_\delta \prec V$ ” stand for the scheme that asserts, of any formula ϕ with a parameter x , that

$$\text{for every } x \in V_\delta, V_\delta \models \phi[x] \text{ if and only if } \phi(x).$$

Lemma 1.16. *Let T be any theory containing ZFC as a subtheory. Then $\text{Con}(T)$ if and only if $\text{Con}(T + V_\delta \prec V)$*

Proof. The implication to the left is trivial. To obtain the implication to the right, let M be a model for T . I will show that, with a suitable interpretation of δ , M is a model for any finite collection of formulae in $T + V_\delta \prec V$, which is therefore consistent. The conclusion then follows by the compactness theorem.

Let Ψ^* be any finite collection of instances of the scheme $V_\delta \prec V$. Let Ψ be the collection of formulas $\psi(x)$, in the language of T , for which there is an instance in Ψ^* of the form “ $\forall x \in V_\delta V_\delta \models \psi[x]$ if and only if $\psi(x)$ ”. We can write $\Psi^* = \{ \text{“}\forall x \in V_\delta V_\delta \models \psi[x] \text{ if and only if } \psi(x)\text{”} \mid \psi \in \Psi \}$.

Ψ is finite, so by Lévy reflection there is an initial segment, M_γ , of M such that for all $\psi \in \Psi$, and for all $x \in M_\gamma$, if $M_\gamma \models \psi[x]$ then $\psi(x)$. So, interpreting δ as γ , $M \models \Psi^*$ (so Ψ^* is consistent). As Ψ^* was an arbitrary

²An alternative approach, not taken here, would be to avoid these nonstandard models by constructing a model for a maximality principle that applies to any finite fragment of ZFC and then finally applying the compactness theorem to find a model of the full maximality principle. Our approach will be to apply the compactness theorem earlier on, and allow iterations to construct models of the full maximality principles.

finite collection of formulas from $V_\delta \prec V$, and since M also satisfies any finite fragment of T (being a model of T), the entire theory $T + V_\delta \prec V$ is therefore consistent. \square

The proof of Lemma 1.16 relies on a compactness argument to establish a model M' for $\text{ZFC} + V_\delta \prec V$ given the existence of a model M for ZFC alone. In fact, a sharper observation can be made which relates these models, namely, it can be arranged that $M \prec M'$. This can be shown by “reproving” the compactness theorem in building M' as an ultraproduct. An alternate proof is also given which follows a Henkin style proof.

We first introduce a notation for an idea that will recur through the chapter. Let T be any theory containing ZFC as a subtheory. Let $I = \{\Phi \subseteq T \mid \Phi \text{ is finite}\}$ be the set of finite collections of formulas in the theory T . For each Φ in I , let $V_\delta \prec_\Phi V$ denote the collection of statements $\{\forall x \in V_\delta (V_\delta \models \phi[x] \text{ if and only if } \phi(x)) \mid \phi \in \Phi\}$, that is, those instances of $V_\delta \prec V$ only mentioning those formulas ϕ in Φ . Notice that any finite subcollection of the scheme $V_\delta \prec V$ can be so represented.

Lemma 1.17. *Let T be any theory containing ZFC as a subtheory. Suppose $M \models T$. Then there is $M' \models T + V_\delta \prec V$ such that $M \prec M'$.*

First proof. Let M be a model for theory T . By the Lévy Reflection Theorem, there is a $\delta = \delta_\Phi$ for which $V_\delta \prec_\Phi V$. So the expansion of the model M to $\langle M, \delta_\Phi \rangle$ is a model of $T + V_\delta \prec_\Phi V$. Denote this expanded model by M_Φ . (Notice that if $\Phi \subseteq \Psi$, then $M_\Psi \models V_\delta \prec_\Phi V$.) Now construct an ultrafilter on I as follows: For each $\Phi \in I$, define $d_\Phi = \{\Psi \in I \mid \Phi \subseteq \Psi\}$, the set of finite collections of formulas containing Φ as a subcollection. Then $D_I = \{d_\Phi \mid \Phi \in I\}$ is easily seen to be a filter on I . And by Zorn’s Lemma, there is an ultrafilter $U_I \supseteq D_I$. We can now take the ultraproduct $\langle M', \delta \rangle = \prod M_\Phi / U_I$. Then $\langle M', \delta \rangle$ is a model of $V_\delta \prec_\Phi V$, for any Φ in I (just apply Łos’s Theorem to the set $\{\Psi \in I \mid M_\Psi \models V_\delta \prec_\Phi V\} \supseteq d_\Phi \in U_I$).

Since $\langle M', \delta \rangle$ is a model of any finite subcollection of $V_\delta \prec V$ (where δ is the element of the ultraproduct which represents the equivalence class of the mapping I to M via $\Phi \mapsto \delta_\Phi$), it must satisfy the entire scheme $V_\delta \prec V$. Finally, the reduct M' of $\langle M', \delta \rangle$ is simply the ultrapower of the model M over the ultrafilter U_I , so $M' \models T$ and $M \prec M'$. \square

Second proof. Let M be a model for theory T . Let T' be the elementary diagram of M together with the scheme $V_\delta \prec V$. By a Lévy Reflection argument,

T' is finitely consistent: any finite set of statements of T' is consistent. So by the compactness theorem, there is a model M' of T' . But any such model is an elementary extension of M , since each element of M interprets its own constant in the elementary diagram which subsequently has an interpretation in M' . \square

The next lemma says that $V_\delta \prec V$ persists over forcing extensions when forcing with forcing notions contained in V_δ . We will refer to such forcing extensions as “small”. (The condition $\mathbb{P} \in V_\delta$ precludes, say, collapsing δ to ω , which would destroy the scheme $V_\delta \prec V$.) In the following results, $V_\delta[G] \prec V[G]$ will mean the obvious thing, namely that $V[G] \models V_\delta \prec V$, where V_δ is interpreted as $V_\delta[G]$ in $V[G]$. The expression $V_\delta[G]$ is unambiguous, since in our usage, G is always generic over small forcing, in which case $(V_\delta)[G] = (V[G])_\delta$.

Lemma 1.18. *Let $V_\delta \prec V$, let $\mathbb{P} \in V_\delta$ be a notion of forcing and let G be V -generic over \mathbb{P} . Then $V_\delta[G] \prec V[G]$.*

Proof. Suppose $x \in V_\delta[G]$ such that $V_\delta[G] \models \phi(x)$. Then there is a condition $p \in G \subseteq \mathbb{P}$ such that $V_\delta \models p \Vdash \phi(\dot{x})$, so by elementarity $V \models p \Vdash \phi(\dot{x})$, hence $V[G] \models \phi(x)$. \square

This is the way an initial segment V_δ , an elementary submodel of the universe, is used in a forcing iteration to obtain a model in which a desired maximality principle holds. Once a forcing notion has been found at each stage to force the necessity of a particular forceably necessary formula in the elementary submodel, a generic G can then be taken to produce an actual extension which is also an elementary submodel in which the next iteration is definable.

Another direction in which Lemma 1.16 can be generalized is to show there is an unbounded class of cardinals δ for which $V_\delta \prec V$. This uses the following strong form of the reflection theorem.

Lemma 1.19 (Lévy). *For any finite list of formulas Φ in the language of ZFC the following is a theorem of ZFC:*

There is a club class of cardinals such that for each δ in C , $V_\delta \prec_\Phi V$.

Proof. Without loss of generality, suppose the list $\Phi = \{\phi_1 \dots \phi_n\}$ is closed under subformulas. Define a class function $f : ORD \rightarrow ORD$ as follows: For α in ORD , let $f(\alpha)$ be the least ordinal γ such that, for all x in V_α ,

and for all $i = 1 \dots n$ there exists y such that $\phi_i(x, y)$ and y is in V_γ . Let $D = \{\beta \in ORD \mid f''\beta \subseteq \beta\}$ = the closure points of f . It is easy to see that D is a club class. For any δ in D , absoluteness for $\phi_1 \dots \phi_n$ between V_δ and V can be proven by induction on the complexity of each ϕ_i : Absoluteness will be preserved under boolean connectives, and the same is true under existential quantification (the Tarski-Vaught criterion is satisfied since δ is a closure point of the function f). Since the cardinals also form a club class, the intersection of them with D will be the desired club C . \square

This leads to the next theorem, another variation of Lemma 1.16. As in Lemma 1.17, the “ground” model is expanded to interpret a new predicate symbol in the language, which in this case is a name for a club class.

Theorem 1.20. *Let T_0 be a theory containing ZFC. Then the following are equivalent:*

- (1) $Con(T_0)$
- (2) $Con(T_0 + T)$

where T is the theory in the language $\{\in, C\}$ which contains all instances of the Replacement and Comprehension axiom schemes augmented with the relation symbol C added to the language, and which asserts

- (i) $C = \langle \delta_\alpha \mid \alpha \in ORD \rangle$ is an unbounded class of cardinals.
- (ii) For any formula ϕ , the following is true: for all δ in C , $V_\delta \prec V$.
- (iii) for all α in ORD , $\delta_\alpha < \text{cof}(\delta_{\alpha+1})$.

Proof. (2) implies (1): Trivial.

(1) implies (2): We first give a model where theory T only includes assertions (i) and (ii). Let M be a model of theory T_0 . It will suffice to show that every finite subtheory of $T_0 + T$ is consistent. So let F be such a finite subtheory. That is, $F \subseteq \text{ZFC} + \{\sigma_1, \dots, \sigma_n\} \cup \{\text{“}C \text{ is club”}\}$, where $\sigma_i = \text{“}\forall \delta \in C \forall x \in V_\delta (V_\delta \models \phi_i[x] \text{ if and only if } \phi_i(x))\text{”}$. We show that M is a model for F : Fix $\{\phi_1, \dots, \phi_n\}$. By Lemma 1.19, there is a definable club class of cardinals C such that for all i in $1, \dots, n$, $M \models \text{“}C \text{ is a club class”} + \text{“}\forall \delta \in C, V_\delta \models \phi_i[x] \text{ if and only if } \phi_i(x)\text{”}$. But this is σ_i , so $M \models \sigma_i$. And since M models T_0 , it is a model for F . And by compactness, M all of

T_0 and T , where T includes assertions (i) and (ii). But from M we can find a model which has a club C which satisfies assertion (iii) as well. Simply define a new class club to be a continuous subsequence of the original one by inductively defining, for each ordinal α , $\delta_{\alpha+1}$ = the δ_α^{+th} element that follows δ_α in C . Take suprema (which are, in fact, unions) at limits. Interpreting the symbol C to be this thinned club ensures that $\delta_\alpha < \text{cof}(\delta_{\alpha+1})$. \square

One should stress that in all applications of Theorem 1.20 one is using the language of ZFC expanded to have the relation symbol C , and that the theory includes theory T as described in the theorem as well as ZFC in this expanded language.

Lemma 1.17 makes it conceptually easier to include in the theory T statements about some element κ of V referring to it by a name added to the language of ZFC. This is done by expanding the model M to interpret the name of κ . Equiconsistency of such statements together with $V_\delta \prec V$ follows from Lemma 1.17 since the name for κ is “rigid”—the model for $V_\delta \prec V$ can be taken to be an elementary extension, so the same κ can be found in both models. The next lemma, illustrating this, is an enhancement of Lemma 1.16 that provides a condition on δ .

Lemma 1.21. *Let T be any theory containing ZFC as a subtheory. Let κ be any ordinal in a model M of T expanded to include κ . Then there is an elementary extension M' of M which is a model of $T + V_\delta \prec V + \text{cof}(\delta) > \kappa$*

Proof. We proceed exactly as in Lemma 1.16, performing additional work to address the requirement on δ :

By Lévy Reflection Lemma 1.19, for any fixed finite $\Phi \subseteq T$, the class $\{\alpha \in \text{ORD} \mid V_\alpha \prec_\Phi V\}$ is closed and unbounded. So it has a κ^{+th} member. We interpret δ as this member, giving $\text{cof}(\delta) = \kappa^+$ and $V_\alpha \prec_\Phi V$. The rest of the proof of Lemma 1.16 now gives $M' \models T + V_\delta \prec V + \text{cof}(\delta) > \kappa$. \square

A typical application of this lemma is to ensure that the cofinality of δ is greater than ω .

We include one equiconsistency result to be used in the next chapter that uses the results of this section.

Lemma 1.22. *Let X be any set, and let Γ be an S_4 class of forcing notions, closed under iterations of length $\kappa = |X|$ with appropriate support. Suppose $V_\delta \prec V$ and $\text{cof}(\delta) > \kappa$. Then there is a forcing notion \mathbb{P} in Γ such that $V^\mathbb{P} \models \text{MP}_\Gamma(X^V)$ and $\mathbb{P} \in V_\delta$.*

Remark. The notation “ X^V ” for the parameter set is to emphasize the *de re* interpretation of the symbol X — that X represents the same set in $V^{\mathbb{P}}$ as when interpreted in V . This is a different situation from Lemma 3.2, where the parameter set \mathbb{R}^V is different from \mathbb{R} in $V^{\mathbb{P}}$. This is because \mathbb{R} is a definition of a set, and is interpreted *de dicto*, hence differently in V and $V^{\mathbb{P}}$.

Proof. Let $\kappa = |X|$. Let $\pi : \omega \times \kappa \longrightarrow \kappa$ be a bijective pairing function. Enumerate all formulas with one parameter in the language of set theory as $\langle \phi_n | n \in \omega \rangle$ and all elements x of X as $\langle x_\mu | \mu \in \kappa \rangle$. Define a κ -iteration $\mathbb{P} = \mathbb{P}_\kappa$ of Γ forcing notions, with appropriate support, as follows. At successor stages, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where, if $\alpha = \pi(n, \mu)$ and $V_\delta^{\mathbb{P}_\alpha} \models “\phi_n(x_\mu) \text{ is } \Gamma\text{-forceably necessary}”$, then $V_\delta^{\mathbb{P}_\alpha} \models “\dot{\mathbb{Q}}_\alpha \text{ is a forcing notion in } \Gamma \text{ that forces ‘}\phi_n(x_\mu) \text{ is } \Gamma\text{-necessary’}”$; otherwise, $\dot{\mathbb{Q}}_\alpha$ is $\{\emptyset\}$, the trivial forcing. Use appropriate support at limit stages. Note that, since $\text{cof}(\delta) > \kappa$, \mathbb{P}_α is in V_δ for all $\alpha < \kappa$.

Let G be V -generic over \mathbb{P} . The claim is that $V[G] \models \text{MP}_\Gamma(X)$. To prove the claim, suppose $x \in X$ and $V[G] \models “\phi(x) \text{ is } \Gamma\text{-forceably necessary}”$. It will suffice to show that $V[G] \models “\phi(x) \text{ is } \Gamma\text{-necessary}”$. Let $\phi = \phi_n$ and $x = x_\mu$ for some $\alpha = \pi(n, \mu)$. By factoring, $V[G] = V[G_\alpha][G_{TAIL}] \models “\phi(x) \text{ is } \Gamma\text{-forceably necessary}”$, so $V[G_\alpha] \models “\phi(x) \text{ is } \Gamma\text{-forceably necessary}”$ as well. By elementarity, $V_\delta[G_\alpha] \models “\phi(x) \text{ is } \Gamma\text{-forceably necessary}”$. But at stage α , the forcing notion $\dot{\mathbb{Q}}$ in V_δ has been defined to force $\Box_\Gamma \phi_n(x_\mu)$. So $V_\delta[G_{\alpha+1}] \models “\phi(x) \text{ is } \Gamma\text{-necessary}”$. Again by elementarity, $V[G_{\alpha+1}] \models “\phi(x) \text{ is } \Gamma\text{-necessary}”$. And since $V[G]$ is a Γ -forcing extension of $V[G_{\alpha+1}]$, $V[G] \models “\phi(x) \text{ is } \Gamma\text{-necessary}”$. This proves the claim. Finally, since the iteration of \mathbb{P} has appropriate support, \mathbb{P} is in Γ . And since $\text{cof}(\delta) > \kappa$, \mathbb{P} is in V_δ . \square

Remark. A clear similarity can be seen between Lemma 1.22 and Lemma 1.10. The key difference is that, in Lemma 1.10, we are establishing a proto-maximality principle for a single formula, for which a definition of the forcing relation is available. Here, in Lemma 1.22, we need a uniform definition of the forcing relation for all formulas, hence the need for the set model V_δ .

Chapter 2

MP_Ψ and variations

In this chapter we explore the properties of maximality principles of the form $\text{MP}_\Psi = \text{MP}_{\Gamma_\Psi}$, where Γ_Ψ is the class of forcing notions preserving the truth of some particular sentence Ψ . To introduce classes of this type, we first discuss one sentence, CH, the continuum hypothesis. Let Γ_{CH} be the class of all forcing notions \mathbb{P} that preserve CH. Thus, if it holds, then \mathbb{P} is in Γ_{CH} if and only if $V^\mathbb{P} \models \text{CH}$. And, if CH does not hold, Γ_{CH} is the class of all forcing. Let MP_{CH} denote the modified maximality principle restricted to Γ_{CH} . We will denote the modal operator Γ_{CH} -necessary by \Box_{CH} and denote Γ_{CH} -forceable by \Diamond_{CH} . As a warmup to more general results, we will first show that the principle MP_{CH} is actually logically equivalent to MP itself together with CH. We show the two directions of implication in two separate theorems, since they each generalize to apply to more general classes of forcing notions.

Lemma 2.1. Γ_{CH} is adequate, and therefore an $S4$ forcing class.

Proof. Clearly trivial forcing preserves CH. And if \mathbb{P} preserves CH and $V^\mathbb{P} \models$ “ \mathbb{Q} preserves CH”, then $\mathbb{P} * \mathbb{Q}$ preserves CH as well. And these facts are Γ_{CH} -necessary, since they are provable in ZFC. \square

Remark. Since the modal logic $S4$ applies to the necessity and possibility operators \Box_{CH} and \Diamond_{CH} , the principle MP_{CH} , or “ $\Diamond_{\text{CH}}\Box_{\text{CH}}\phi$ implies ϕ ”, is equivalent to “ $\Diamond_{\text{CH}}\Box_{\text{CH}}\phi$ implies $\Box_{\text{CH}}\phi$ ”. So for the rest of this section the form of the maximality principle we will use in proofs will be

$$\Diamond_{\text{CH}}\Box_{\text{CH}}\phi \text{ implies } \phi,$$

that is, a formula which is Γ_{CH} -forceably necessary must be true.

Theorem 2.2. *The principle MP_{CH} is equivalent to $\text{MP} + \text{CH}$.*

Proof. To prove the forward implication, assume MP_{CH} ($\Diamond_{\text{CH}}\Box_{\text{CH}}\phi$ implies ϕ , for all statements ϕ). We first claim that, in fact, CH must be true: CH is certainly Γ_{CH} -forceable ($\Diamond_{\text{CH}}\text{CH}$). And in the extension where CH becomes true, CH is Γ_{CH} -necessary. So CH is Γ_{CH} -forceably necessary ($\Diamond_{\text{CH}}\Box_{\text{CH}}\text{CH}$), so by MP_{CH} , CH is true.

To establish MP, suppose that ϕ is forceably necessary. It will suffice to infer that ϕ is true. There exists a forcing notion \mathbb{P} such that $V^{\mathbb{P}}$ satisfies that ϕ is necessary. Let \mathbb{Q} be any forcing notion that forces CH to be true in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. Notice that ϕ is necessary in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ as well, since $V^{\mathbb{P}} \models \Box\phi$. So $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ satisfies both $\Box\phi$ and CH. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ_{CH} , we have that $\Box\phi$ is Γ_{CH} -forceable. But $\Box\phi$ implies $\Box_{\text{CH}}\phi$, so $\Diamond_{\text{CH}}\Box_{\text{CH}}\phi$. Finally, by MP_{CH} , ϕ is true.

To prove the reverse implication, suppose ϕ is Γ_{CH} -forceably necessary, that is, $\Box_{\text{CH}}\phi$ is Γ_{CH} -forceable. $\Box_{\text{CH}}\phi$ is therefore forceable. Said another way, there is a forcing extension in which any further forcing satisfies ϕ , unless it is a forcing that is not in Γ_{CH} and does not preserve CH. So $\phi \vee \neg\text{CH}$ is forceably necessary, and by MP, it is true. But CH is true by hypothesis, so ϕ is true, showing that MP_{CH} holds. \square

Theorem 2.3. *Let X be any set. The principle $\text{MP}_{\text{CH}}(X)$ is equivalent to $\text{MP}(X) + \text{CH}$.*

Proof. The same as the previous proof, since whether the formula instances of MP_{CH} used in that proof had parameters or not was irrelevant. \square

Corollary 2.4. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC} + \text{MP}_{\text{CH}})$
- (2) $\text{Con}(\text{ZFC} + \text{MP})$
- (3) $\text{Con}(\text{ZFC})$.

Remark. This answers a question posed by Hamkins in [HAM1].

Proof. (2) \iff (3) by [HAM1]. Since neither CH nor its negation is forceably necessary by general forcing, $\text{ZFC} + \text{MP} + \text{CH}$ is equiconsistent with $\text{ZFC} + \text{MP}$. Then (1) \iff (2) by Theorem 2.2. \square

The preceding result can be generalized by abstracting the essential properties of the classes of forcing notions just discussed. If Γ_1 and Γ_2 are each classes of forcing notions, when can we say that MP_{Γ_1} implies MP_{Γ_2} ? The preceding result suggests possible answers. Suppose Γ_1 and Γ_2 are classes of forcing notions. Let us say that Γ_2 **absorbs**¹ Γ_1 if it is true and Γ_1 -necessary that Γ_2 is contained in Γ_1 and for any \mathbb{P} in Γ_1 there exists $\dot{\mathbb{Q}}$ such that $V^\mathbb{P} \models$ “ $\dot{\mathbb{Q}}$ is in Γ_2 ” and “ $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ_2 ”. One example of this just seen is the following.

Theorem 2.5. Γ_{CH} absorbs the class of all forcing notions.

Proof. Let Γ = all forcing notions. Clearly “ $\Gamma_{\text{CH}} \subseteq \Gamma$ ” is Γ -necessary. Let \mathbb{P} be in Γ . One can find $\dot{\mathbb{Q}}$ in $V^\mathbb{P}$ to force CH, so $\dot{\mathbb{Q}}$ is in Γ_{CH} . And since $\mathbb{P} * \dot{\mathbb{Q}}$ forces CH, $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ_{CH} . \square

Remark. Another example of a class of forcing notions that behaves as Γ_{CH} does in Theorem 2.5 is COLL. The forcing notion $\text{coll}(\omega, \theta)$ is the forcing notion that collapses θ to ω . The forcing class COLL consists of $\text{coll}(\omega, \theta)$, for all $\theta > \omega$. Lemma 5.1 will show that COLL absorbs the class of all forcing notions.

Lemma 2.6 (Absorption Lemma). Suppose Γ_1 and Γ_2 are classes of forcing notions such that Γ_2 absorbs Γ_1 , and ϕ is any formula, with arbitrary set parameters for ϕ . If ϕ is Γ_1 -forceably necessary then it is Γ_2 -forceably necessary as well.

Remark. We allow arbitrary set parameters for ϕ , since this does not affect the argument.

Proof. Assume that ϕ is Γ_1 -forceably necessary. Then there is \mathbb{P} in Γ_1 such that $V^\mathbb{P} \models$ “ ϕ is Γ_1 -necessary”. By absorption, there exists $\dot{\mathbb{Q}}$ such that $V^\mathbb{P} \models$ “ $\dot{\mathbb{Q}}$ is in Γ_2 ” and $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ_2 . Since $V^\mathbb{P} \models \Gamma_2 \subseteq \Gamma_1$, $V^\mathbb{P} \models$ “ $\dot{\mathbb{Q}}$ is in Γ_1 and ϕ is Γ_1 -necessary”. So $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “ ϕ is Γ_1 -necessary”. And since $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “ Γ_2 is contained in Γ_1 ”, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “ $\Box_{\Gamma_1} \phi$ implies $\Box_{\Gamma_2} \phi$ ”. So $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “ ϕ is Γ_2 -necessary”. This establishes that ϕ is Γ_2 -forceably necessary. \square

Corollary 2.7. Let Γ_1 and Γ_2 be S_4 classes of forcing notions. If Γ_2 absorbs Γ_1 then MP_{Γ_2} implies MP_{Γ_1} .

¹The name is vaguely inspired by the absorption law of propositional calculus. Another way to express the relationship between Γ_2 and Γ_1 is to say that Γ_2 is cofinal in Γ_1 where all forcing notions are ordered so that $\mathbb{P} \leq \mathbb{P} * \dot{\mathbb{Q}}$ for any $\dot{\mathbb{Q}}$.

Proof. Assume MP_{Γ_2} (in the S4 form $\Diamond_{\Gamma_2}\Box_{\Gamma_2}\phi$ implies ϕ). Next suppose ϕ is Γ_1 -forceably necessary, with the goal of proving the truth of ϕ . By absorption and Lemma 2.6, ϕ is Γ_2 -forceably necessary. This gives the truth of ϕ by MP_{Γ_2} . \square

Corollary 2.8. *Let Γ_1 and Γ_2 be S4 classes of forcing notions. Let X be any set. If Γ_2 absorbs Γ_1 then $\text{MP}_{\Gamma_2}(X)$ implies $\text{MP}_{\Gamma_1}(X)$.*

Proof. Since Lemma 2.6 allows formulas with arbitrary parameters, just replace ϕ in the proof of Corollary 2.7 with $\phi(x)$, where x is in X . \square

Corollary 2.7 corresponds exactly to how we proved that MP_{CH} implies MP in Theorem 2.2, where Γ_{CH} absorbed the class of all forcing notions.

To generalize the reverse direction of Theorem 2.2, one generalizes the CH-preserving property. Let Ψ be a sentence of the language of ZFC. A forcing notion \mathbb{P} is **Ψ -preserving** if Ψ implies that it is true in $V^{\mathbb{P}}$. I will denote the class of Ψ -preserving forcing notions by Γ_Ψ . The associated maximality principle is MP_Ψ .

Lemma 2.9. *Let Ψ be a sentence of the language of ZFC. Γ_Ψ is S4.*

Proof. This proof emulates the proof of Lemma 2.1.

Clearly trivial forcing preserves Ψ . And if \mathbb{P} preserves Ψ and $V^{\mathbb{P}} \models \text{“}\dot{\mathbb{Q}} \text{ preserves } \Psi\text{”}$, then $\mathbb{P} * \dot{\mathbb{Q}}$ preserves Ψ as well. And these facts are Γ_Ψ -necessary, since they are provable in ZFC. \square

Lemma 2.10. *Let Ψ be a forceable sentence of the language of ZFC. The principle MP_Ψ implies Ψ .*

Proof. If $\neg\Psi$ then Γ_Ψ is all forcing. Since Ψ is forceable, $\Diamond_\Psi\Psi$. And in the extension where Ψ becomes true, Ψ is Γ_Ψ -necessary. So $\Diamond_\Psi\Box_\Psi\Psi$, so by MP_Ψ , Ψ is true. \square

Theorem 2.11. *Let Ψ be a forceable sentence of the language of ZFC. Let Γ be an S4 class of forcing notions. If $\Gamma_\Psi \subseteq \Gamma$ is Γ -necessary then $\text{MP}_\Gamma + \Psi$ implies MP_Ψ .*

Proof. Assume MP_Γ . By S4 we can write it in the form that says $\Diamond_\Gamma\Box_\Gamma\phi$ implies ϕ . Next suppose ϕ is Γ_Ψ -forceably necessary, with the goal of proving ϕ from this. Since $\Gamma_\Psi \subseteq \Gamma$, we have ϕ is Γ -forceably Γ_Ψ -necessary. But $\Box_{\Gamma_\Psi}\phi$ implies $\Box_\Gamma(\phi \vee \neg\Psi)$, since any forcing notion in Γ that is not in Γ_Ψ must

produce a model in which Ψ is false. Combining the last two sentences gives $\Diamond_\Gamma \Box_\Gamma (\phi \vee \neg \Psi)$. Now we apply MP_Γ to get $\phi \vee \neg \Psi$. But Ψ is true by hypothesis, hence ϕ is true. \square

The concept that unites both directions of implication is resurrectibility. A sentence Ψ of the language of ZFC is Γ -**resurrectible** if there is Γ -necessarily a forcing notion $\mathbb{P} \in \Gamma$ such that $V^\mathbb{P} \models \Psi$, in other words, if $\Box_\Gamma \Diamond_\Gamma \Psi$ is true.

Remark. CH is resurrectible—it can be forced to be true over any model of ZFC.

Theorem 2.12. *Let Γ be an S_4 class of forcing notions. If Ψ is a Γ -resurrectible sentence of the language of ZFC and $\Gamma_\Psi \subseteq \Gamma$ is Γ -necessary, then $\text{MP}_\Psi + \Psi \iff \text{MP}_\Gamma + \Psi$.*

Proof. Clearly Γ_Ψ absorbs Γ , since any forcing notion that resurrects Ψ must be in Γ_Ψ . Corollary 2.7 then gives the forward implication. The converse follows directly from Theorem 2.11. \square

This theorem can be applied to obtain useful equiconsistency results.

The next lemma provides some basic facts.

Lemma 2.13. *Let ϕ , ϕ_1 , and ϕ_2 be sentences in the language of ZFC.*

- (1) *If ϕ_1 implies ϕ_2 , then $\Gamma_{\phi_1} \subseteq \Gamma_{\phi_2}$*
- (2) *If ϕ is provable in ZFC, then ϕ is Γ -necessary, for any class Γ of forcing notions.*
- (3) *If ϕ is provably Γ -forceable, then ϕ is Γ -resurrectible.*

Proof. (1) and (2) are obvious (in modal logic terms, (2) is just the necessitation principle), and (3) follows from (2). \square

Another basic fact is

Lemma 2.14. *If Ψ is provably Γ -forceable, then $\text{Con}(\text{ZFC} + \text{MP}_\Gamma)$ if and only if $\text{Con}(\text{ZFC} + \text{MP}_\Gamma + \Psi)$*

Proof. Let $M \models \text{MP}_\Gamma + \text{ZFC}$. Let $M[G]$ be a Γ -forcing extension of M such that $M[G] \models \Psi$. But then $M[G] \models \text{MP}_\Gamma$ as well: If $M[G] \models \text{“}\phi \text{ is } \Gamma\text{-forceably necessary”}$ then $M \models \text{“}\phi \text{ is } \Gamma\text{-forceably necessary”}$, so $M \models \text{“}\phi \text{ is } \Gamma\text{-necessary”}$ and therefore $M[G] \models \text{“}\phi \text{ is } \Gamma\text{-necessary”}$. \square

Recall that Suslin trees, which are provably forceable either by Cohen forcing or $< \omega_1$ -closed forcing, negate Suslin's Hypothesis by their existence. By the above facts, the sentence $\neg SH$ is resurrectible in the class of all forcing notions. (In fact, any resurrectible sentence could be used in the following theorem.)

Theorem 2.15. *The following are equivalent:*

- (1) $Con(ZFC + MP_{\neg SH})$
- (2) $Con(ZFC + MP)$
- (3) $Con(ZFC)$.

Proof. (1) \Leftrightarrow (2): Via Theorem 2.12 and Lemma 2.14.

(2) \Leftrightarrow (3): Via [HAM1]. □

Chapter 3

MP_{CCC} and variations

3.1 The consistency strength of $\text{MP}_{\text{CCC}}(\mathbb{R})$

In this section we show a surprising result. [HAM1] shows that the maximality principle with real parameters, $\text{MP}(\mathbb{R})$, has consistency strength strictly greater than ZFC, while MP and MP_{CCC} are both equiconsistent with ZFC. So it would seem that adding the parameter set \mathbb{R} should increase the consistency strength of MP_{CCC} as it did for MP , especially since CCC-forcing will certainly add new reals that are not in the ground model. However, contrary to expectation, we have the following theorem:

Theorem 3.1. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC})$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{CCC}}(\mathbb{R}))$

Proof. (2) implies (1): trivial.

(1) implies (2): It will suffice to show that, given a model of ZFC, one can produce a model of $\text{ZFC} + \text{MP}_{\text{CCC}}(\mathbb{R})$. I will give two proofs of this.

To begin the first proof, we prove the consistency of a weak version of this principle. For any set X , using our notation, $\text{MP}_{\text{CCC}}(X)$ is the modified maximality principle that says any formula with parameters taken from the set X which is CCC-forceably necessary is true. Let \mathbb{P} be a CCC forcing notion. Let us confine ourselves to the model $V^{\mathbb{P}}$, and denote by \mathbb{R}^V the set of reals of $V^{\mathbb{P}}$ which are not introduced by forcing with \mathbb{P} , that is, all reals of the ground model V . Let the principle $\text{MP}_{\text{CCC}}(\mathbb{R}^V)$ be the form of $\text{MP}_{\text{CCC}}(X)$ interpreted in $V^{\mathbb{P}}$ with parameter set \mathbb{R}^V .

Lemma 3.2. *Suppose $V_\delta \prec V$, and $\text{cof}(\delta) > 2^\omega$. Then there is a forcing notion \mathbb{P} in CCC such that $V^\mathbb{P} \models \text{MP}_{\text{CCC}}(\mathbb{R}^V)$ and $\mathbb{P} \in V_\delta$.*

Proof. This is an instance of Lemma 1.22. \square

Lemma 3.3. *If there is a model of ZFC then there is a model of ZFC + $\text{MP}_{\text{CCC}}(\mathbb{R})$.*

First proof. Suppose $V \models \text{ZFC}$. We will construct a forcing extension which is a model of ZFC + $\text{MP}_{\text{CCC}}(\mathbb{R})$. By Theorem 1.20 we may assume that there is a class club of cardinals C such that for all δ in C , $V_\delta \prec V$. Construct a finite-support ω_1 -iteration $\mathbb{P} = \mathbb{P}_{\omega_1}$, such that $V^\mathbb{P} \models \text{MP}_{\text{CCC}}(\mathbb{R})$, as follows. Let \mathbb{P}_0 be the trivial notion of forcing. At stage α , select δ_α from the club C such that the rank of $\mathbb{P}_\alpha < \delta_\alpha$ (so that \mathbb{P}_α is in V_{δ_α}) and $\text{cof}(\delta_\alpha) > (2^\omega)^{V^\mathbb{P}_\alpha}$. Working in $V^\mathbb{P}_\alpha$, define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where \mathbb{Q} is a \mathbb{P}_α -name of a CCC notion of forcing such that $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models \text{MP}_{\text{CCC}}(\mathbb{R}^{V^\mathbb{P}_\alpha})$. Such a \mathbb{Q}_α is guaranteed to exist by Lemma 3.2, since the conditions $V_{\delta_\alpha} \prec V$ and $\text{cof}(\delta_\alpha) > 2^\omega$, are satisfied by $V^\mathbb{P}_\alpha$. This completes the construction of $\mathbb{P} = \mathbb{P}_{\omega_1}$.

Let G be V -generic over \mathbb{P} . I claim that $V[G] \models \text{MP}_{\text{CCC}}(\mathbb{R})$. To see this, let $V[G] \models \text{"}\phi(r) \text{ is CCC-forceably necessary"}$, where $r \in \mathbb{R}$, the reals as interpreted in $V[G]$. It will suffice to show that $V[G] \models \text{"}\phi(r) \text{ is CCC-necessary"}$. Note that r , as a real, is a subset of ω in $V[G]$, while ω itself is in V . Therefore, since \mathbb{P} is an ω_1 iteration with finite support and $\text{cof}(\omega_1) > |\omega|$, r must be in some $V[G_\alpha]$, where $\alpha < \omega_1$, $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{P}}_{\omega_1}^{(\alpha)}$ is the factorization of \mathbb{P} at stage α , and G_α is the projection of G to \mathbb{P}_α . (This follows from [KUN], Chapter VIII, Lemma 5.14.) But the definition of \mathbb{P} required that \mathbb{Q}_α force $\text{MP}_{\text{CCC}}(\mathbb{R}^{V[G_\alpha]})$, which therefore must hold at stage $\alpha + 1$. Indeed, refactoring $\mathbb{P} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}_{\omega_1}^{(\alpha+1)}$ and setting $G_{\alpha+1}$ to be the projection of G to $\mathbb{P}_{\alpha+1}$, we have that $r \in V[G_\alpha]$ and $V[G_{\alpha+1}] \models \text{MP}_{\text{CCC}}(\mathbb{R}^{V[G_\alpha]})$ (as well as $V[G_{\alpha+1}] \models \text{"}\phi(r) \text{ is CCC-forceably necessary"}$, since $V[G] = V[G_{\alpha+1}][G_{\omega_1}^{(\alpha+1)}]$ is a CCC-forcing extension of $V[G_{\alpha+1}]$). Therefore $V[G_{\alpha+1}] \models \text{"}\phi(r) \text{ is CCC-necessary"}$. Since $V[G]$ is a CCC-forcing extension of $V[G_{\alpha+1}]$, we have that $V[G] \models \text{"}\phi(r) \text{ is CCC-necessary"}$, as required. \square

Second proof. This time, we use a bookkeeping function style argument. Again, suppose $V \models \text{ZFC}$. By Theorem 1.20 we may assume that there is in V a class club of cardinals C such that for all δ in C , $V_\delta \prec V$. Let $\pi : \text{ORD} \simeq \omega \times \text{ORD} \times \text{ORD}$ be a definable bijective class function $\pi : \alpha \mapsto \langle n, \beta, \mu \rangle$ such that $\beta \leq \alpha$. Using π as a bookkeeping function,

we define a sequence of iterated forcing notions \mathbb{P}_α , simultaneously with a sequence of cardinals δ_α by transfinite induction on α in ORD , as follows. Let \mathbb{P}_0 be trivial forcing. Given \mathbb{P}_α , let δ_α be the least cardinal in the club C such that \mathbb{P}_α is in V_{δ_α} . Define $\dot{\mathbb{Q}}_\alpha$ in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$ as follows: Let $\pi(\alpha) = \langle n, \beta, \mu \rangle$. Consider the statement $\phi(x) = \phi_n(x)$, the n^{th} statement in the language of ZFC according to some enumeration, with single parameter $x = x_\mu$, the μ^{th} name for a real in the model $V^{\mathbb{P}_\beta}$ where $\beta \leq \alpha$.¹ If, in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$, $\phi(x)$ is CCC-forceably necessary, let $\dot{\mathbb{Q}}_\alpha$ be the V_{δ_α} -least \mathbb{P}_α -name of a forcing notion which performs a forcing that $\phi(x)$ is CCC-necessary. Otherwise let $\dot{\mathbb{Q}}_\alpha$ be the \mathbb{P}_α -name for trivial forcing. Now let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Finally, take finite support at limits. This defines the sequence \mathbb{P}_α for all α in ORD . Note that, for all such α , \mathbb{P}_α is CCC and is contained in V_{δ_α} .

We wish to truncate this sequence at an appropriate length λ to obtain an iterated forcing notion \mathbb{P}_λ which forces a model of $\text{MP}_{\text{CCC}}(\mathbb{R})$. This will occur if all reals in $V^{\mathbb{P}_\lambda}$ are introduced at some earlier stage of the iteration and the cofinality of λ is greater than ω . To ensure this, we define λ to be a closure point of the function $f : ORD \rightarrow ORD$ which takes β , the stage at which a real parameter is introduced, to the least stage by which all formulas ϕ have been applied to all parameters in $V^{\mathbb{P}_\beta}$. We now make use of the technique of using *nice names*, discussed in detail in [KUN]. Since we only need to count nice names, of which there are $|\mathbb{P}_\beta|^\omega$ many in $V^{\mathbb{P}_\beta}$, this gives $f(\beta) = \sup_{\mu < |\mathbb{P}_\beta|^\omega} \{\pi(\alpha) = \langle n, \beta, \mu \rangle\}$. Now let λ be the first closure point of $f : ORD \rightarrow ORD$ with cofinality ω_1 . (An ordinal α is a **closure point** of f if $f''\alpha \subseteq \alpha$.)

Let $\mathbb{P} = \mathbb{P}_\lambda$, and let G be V -generic over \mathbb{P} . By the usual argument, we can now establish that $V[G] \models \text{MP}_{\text{CCC}}(\mathbb{R})$: Suppose $V[G] \models \text{"}\phi(r) \text{ is CCC-forceably necessary"}$. Then there is $\alpha = \langle n, \mu, \beta \rangle$, where $\phi = \phi_n$ and \dot{r} is the μ^{th} nice \mathbb{P}_β -name of a real, for some $\beta \leq \alpha$ (the name \dot{r} appears before stage δ by the reasoning used in Lemma 1.22). Since $V[G]$ is a CCC-forcing extension of $V[G_\alpha]$, where G_α is V -generic over \mathbb{P}_α , $V[G_\alpha] \models \text{"}\phi(r) \text{ is CCC-forceably necessary"}$, whence by elementarity, $V_{\delta_\alpha}[G_\alpha] \models \text{"}\phi(r) \text{ is CCC-forceably necessary"}$. But by the construction of \mathbb{P} , if $G_{\alpha+1}$ is $\mathbb{P}_{\alpha+1}$ -generic over $V_{\delta_{\alpha+1}}$ then $V_{\delta_{\alpha+1}}[G_{\alpha+1}] \models \text{"}\phi(r) \text{ is CCC-necessary"}$, so by elementarity $V[G_{\alpha+1}] \models \text{"}\phi(r) \text{ is CCC-necessary"}$ and therefore $V[G] \models \text{"}\phi(r) \text{ is CCC-}$

¹Assuming an enumeration of the names for reals in $V^{\mathbb{P}_\beta}$ is based on a wellordering of the ground model, which can be forced by class forcing that adds no new sets. This induces a wellordering of names, giving a wellordering of any forcing extension.

necessary". □

This concludes the proof of Theorem 3.1. □

Remark. This raises the possibility that MP_{CCC} directly implies $\text{MP}_{\text{CCC}}(\mathbb{R})$. But we will see that this is false.

Remark. The proof just given of Lemma 3.3 makes use of the existence of closure points of the defined function $f : \text{ORD} \rightarrow \text{ORD}$. In order to know such closure points exist one needs to apply the Replacement scheme. Even the first proof makes use of the Replacement Axiom Scheme enhanced with the symbol C , in order to construct the iteration \mathbb{P}_α . These arguments take place in the language of ZFC expanded with the symbol C interpreted as a class club in an expanded model. This is why we included, in Theorem 1.20, all instances of Replacement and Comprehension that mention the class club C .

One might expect that Theorem 3.1 can be extended to parameter sets which are power sets of sets of cardinality greater than ω , such as ω_1 or $\aleph_{\omega_{17}}$, and in fact this is the case. Let κ be a cardinal. Singling out the second proof strategy above, one can state and prove the generalizations of Lemma 3.3 and Theorem 3.1 as:

Lemma 3.4. *Let κ be any CCC-absolutely definable cardinal. If there is a model of ZFC then there is a model of $\text{ZFC} + \text{MP}_{\text{CCC}}(H(\kappa))$*

Proof. As before, the proof consists in finding a CCC-forcing extension model for $\text{MP}_{\text{CCC}}(H(\kappa))$. Let V be a model of ZFC. We use the same bookkeeping class function and again assume in V a class club of cardinals C such that for all δ in C , $V_\delta \prec V$. The definition of a sequence of iterated forcing notions \mathbb{P}_α , simultaneously with a sequence of cardinals δ_α again proceeds by transfinite induction on α in ORD . At stage α , in defining \dot{Q}_α in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$, α now codes $\langle n, \beta, \mu \rangle$ where μ is now an index for the name for a subset x of κ in the model $V^{\mathbb{P}_\beta}$ where $\beta \leq \alpha$, and \dot{Q}_α forces that $\phi(x)$ is CCC-necessary if such a forcing notion exists. Again let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$ with finite support at limits. This again gives \mathbb{P}_α for all α in ORD , with \mathbb{P}_α being CCC and contained in V_{δ_α} .

The length λ of the iteration giving a \mathbb{P}_λ which satisfies $\text{MP}_{\text{CCC}}(H(\kappa))$ must now have cofinality κ , to ensure that all names of parameters in $V^{\mathbb{P}_\lambda}$ are introduced at earlier stages. It must also be a closure point of the function $f : \text{ORD} \rightarrow \text{ORD}$ which takes β , the stage at which a real parameter is

introduced, to the least stage by which all formulas ϕ have been applied to all parameters in $V^{\mathbb{P}_\beta}$. Since we only need to count nice names, of which there are $(|\mathbb{P}_\beta|^\omega)^{<\kappa} = |\mathbb{P}_\beta|^{<\kappa}$ many in $V^{\mathbb{P}_\beta}$, this gives $f(\beta) = \sup_{\mu < |\mathbb{P}_\beta|^{<\kappa}} \{\alpha = \langle n, \beta, \mu \rangle\}$.

Let $\mathbb{P} = \mathbb{P}_\lambda$, and let G be V -generic over \mathbb{P} . By an analogous argument, $V[G] \models \text{MP}_{\text{CCC}}(H(\kappa))$. Since $\text{cof}(\lambda) > \kappa$, all parameters in $V[G]$ appear at some previous stage. The rest of the argument is identical to that for Lemma 3.3. \square

Theorem 3.5. *Let κ be any CCC-absolutely definable cardinal. Then the following are equivalent:*

- (1) $\text{Con}(\text{ZFC})$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{CCC}}(H(\kappa)))$

Proof. Immediate from Lemma 3.4 \square

Remark. This raises the possibility that MP_{CCC} directly implies $\text{MP}_{\text{CCC}}(H(\kappa))$ for any CCC-absolutely definable cardinal κ . But we will see that this is false.

Finally, we prove what appears to be an optimal result in this direction. It is optimal in the sense that $\text{MP}_{\text{CCC}}(H(2^\omega))$ is known to be equiconsistent with an inaccessible cardinal, as proved in Lemma 5.9 in [HAM1], and the following proof is a straightforward modification of that lemma by only dropping the weak inaccessibility (i.e. regularity) of $\delta = 2^\omega$.

Theorem 3.6. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC})$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{CCC}}(H(\text{cof}(2^\omega))))$

Proof. That (2) implies (1) is trivial. For the converse suppose without loss of generality that $V_\delta \prec V$, since this is equiconsistent with ZFC. Define a bijective bookkeeping function $\delta \rightarrow \omega \times \delta \times \delta$ such that if $\alpha = \langle n, \mu, \beta \rangle$ then $\beta < \alpha$. Enumerate the formulas having one free variable in the language of ZFC as $\langle \phi_n | n \in \omega \rangle$. Define a finite support δ -iteration of CCC forcing $\mathbb{P} = \mathbb{P}_\delta$ as follows. Let \mathbb{P}_0 be trivial forcing. Given \mathbb{P}_α , let $\alpha = \langle n, \mu, \beta \rangle$. If $\phi_n(x)$ is CCC-forceably necessary in $V_\delta^{\mathbb{P}_\alpha}$ with x the μ^{th} \mathbb{P}_β -name of a parameter from $H(\text{cof}(2^\omega))$ in V_δ , then let \mathbb{Q}_α be a CCC forcing notion such that $V_\delta^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models \text{"}\phi_n(x)\text{" is CCC-necessary}$. Otherwise let \mathbb{Q}_α be trivial forcing.

Let $G \subseteq \mathbb{P}_\delta$ be V -generic for \mathbb{P}_δ . We claim $V[G] \models \text{MP}_{\text{CCC}}(H(\text{cof}(\delta)))$. Also, the CCC iteration \mathbb{P}_δ preserves the cofinality of δ .

Now let $V[G] \models \text{“}\phi(x) \text{ is CCC-forceably necessary”}$, where x is in $H(\text{cof}(\delta))$. It suffices to show that $V[G] \models \text{“}\phi(x) \text{ is CCC-necessary”}$. Since x is in $H(\text{cof}(\delta))$, it appears at some stage of the iteration, \mathbb{P}_β , as the μ^{th} parameter from $H(\delta)$ in $V^{\mathbb{P}_\beta}$, and $\phi = \phi_n$ for some n . Let $\alpha = \langle n, \mu, \beta \rangle$ be less than δ . Then $V[G]$ is a CCC-extension of $V[G_\alpha]$, where G_α is V -generic over \mathbb{P}_α . So $V[G_\alpha] \models \text{“}\phi(x) \text{ is CCC-forceably necessary”}$. By elementarity, $V_\delta[G_\alpha] \models \text{“}\phi(x) \text{ is CCC-forceably necessary”}$. But by the construction of \mathbb{P}_δ , $V_\delta[G_{\alpha+1}] \models \text{“}\phi(x) \text{ is CCC-necessary”}$, where $G_{\alpha+1}$ is V -generic over $\mathbb{P}_{\alpha+1}$. By elementarity, $V[G_{\alpha+1}] \models \text{“}\phi(x) \text{ is CCC-necessary”}$. And since $V[G]$ is a CCC-extension of $V[G_{\alpha+1}]$, $V[G] \models \text{“}\phi(x) \text{ is CCC-necessary”}$.

Finally, the CCC iterations below stage δ will leave 2^ω less than δ (being within V_δ , as an elementary submodel of V) but otherwise arbitrarily large (for any given $\kappa < \delta$, it is CCC-forceably necessary that $2^\omega > \kappa$). So $V[G] \models 2^\omega \geq \delta$. But forcing of size $\leq \delta$ imposes the restriction $V[G] \models 2^\omega \leq \delta$. So $V[G] \models 2^\omega = \delta$, and therefore $V[G]$ models $\text{MP}_{\text{CCC}}(H(\text{cof}(2^\omega)))$. \square

Remark. Notice that, by modifying the proof to invoke Theorem 1.20, we can make $\text{cof}(\delta)$ as large as we like. Let's call the desired cofinality κ . Just choose δ from the class club C to have the desired cofinality, by picking the κ^{th} element in C . For example, if $\text{cof}(\delta)$ is chosen to be ω_{23} , we get a model of $\text{ZFC} + \text{MP}_{\text{CCC}}(H(\omega_{23})) + \text{cof}(2^\omega) = \omega_{23}$, an improvement over Theorem 3.5.

Remark. The construction in the proof of this theorem actually provides a stronger parameter set than $H(\text{cof}(2^\omega))$. At each stage α , all parameters from $H(\delta)$ are allowed, as seen from $V^{\mathbb{P}_\alpha}$. Notice that such stages exist for all $\alpha < \delta = 2^\omega$, so if η is an ordinal below α , it is seen by $V^{\mathbb{P}_\beta}$ as being in $H(\delta)$, for $\beta < \alpha$. Since all ordinals below $\delta = 2^\omega$ eventually achieve this status, the set of ordinals below 2^ω , i.e., the set 2^ω , can be used as a parameter set. So the actual result of the theorem is $\text{Con}(\text{ZFC}) \iff \text{Con}(\text{ZFC} + \text{MP}_{\text{CCC}}(H(\text{cof}(2^\omega) \cup 2^\omega)))$.

Remark. A similar result is given as theorem 3.9 in [SV], but there is a subtle difference between that theorem and Theorem 3.6. Stavi and Väänänen discuss maximality principles with various parameter sets including $H(2^\omega)$ and $H(\text{cof}(2^\omega))$, as well as various classes of forcing notions, including CCC and Cohen forcing. Their statements of maximality principles for a class Γ apply to statements in the language of ZFC which are provably Γ -persistent. (A statement ϕ is Γ -persistent if when ϕ is true, then it is true in all Γ -forcing

extensions.) They also seem to have provided the first proofs of maximality principles by iterating over all instances. The lineage of such proofs, tracing citations, goes from [SV] through [ASP] to [HAM1].

3.2 $\Box\text{MP}_{\text{CCC}}$

One certainly expects the consistency strength of $\Box\text{MP}_{\text{CCC}}$ with parameters to be significantly higher than MP_{CCC} . Again we can informally argue this case by making analogies with MP and $\Box\text{MP}(\mathbb{R})$, the consistency strength of the latter being at least that of the Axiom of Determinacy, according to a result of Woodin (see [HAM1]). We recall the point made in the introduction that an uncountable parameter is not acceptable for the principle MP (since the assertion of its uncountability will be falsified by the principle since its cardinality can be collapsed to ω). With CCC forcing this problem does not occur, so it seems larger parameter sets should be acceptable. But when we attempt to include uncountable parameters by using the parameter set $H(\omega_2)$, the result is unexpected.

Theorem 3.7. *Let Γ be a class of forcing notions necessarily containing CCC. Then $\Box\text{MP}_\Gamma(H(\omega_2))$ is false.*

To prove this theorem, we will utilize some well-known connections among CCC forcing, Martin's Axiom (MA), and Suslin trees. Recall that an (ω_1) -Suslin tree is an ω_1 -tree which has no uncountable chain or antichain.

Lemma 3.8. *Let Γ be a class of forcing notions necessarily containing CCC. Then $\text{MP}_\Gamma(H(\omega_2))$ implies MA_{ω_1} .*

Proof. Recall that MA_{ω_1} is the statement that for any forcing notion \mathbb{P} which is CCC, and any family \mathcal{D} of sets dense in \mathbb{P} , where $|\mathcal{D}| = \omega_1$, there is a filter G meeting each $D \in \mathcal{D}$. It is a fact [KUN] that to establish MA_{ω_1} it suffices to consider forcing notions of size $\leq \omega_1$. Assume \mathbb{P} is a CCC forcing notion such that $|\mathbb{P}| < \omega_1$. Assume also that \mathcal{D} is a family of dense sets in \mathbb{P} where $|\mathcal{D}| = \omega_1$. Let $\phi(p, d)$ be the assertion that there is a filter G in the forcing notion p meeting each D in d . $\phi(\mathbb{P}, \mathcal{D})$ is included in the scope of this principle, since the two parameters can be regarded as a single parameter which is in $H(\omega_2)$. (It is an ordered pair whose components are each of size ω_1 .)

Claim 1. $\phi(\mathbb{P}, \mathcal{D})$ is Γ -forceably necessary.

Proof. We will find a Γ -forcing extension (in fact, a CCC-forcing extension) in which $\phi(\mathbb{P}, \mathcal{D})$ is necessary. We do this by regarding \mathbb{P} as a forcing notion where p extends q iff $p \leq q$ in \mathbb{P} ; it satisfies the countable chain condition, so it is in Γ . So in the resulting model $V^{\mathbb{P}}$, there is a generic filter $G \subseteq \mathbb{P}$ that will meet all dense sets in \mathbb{P} , a fortiori meeting each $D \in \mathcal{D}$. So $\phi(\mathbb{P}, \mathcal{D})$ is Γ -forceable, since it is true in $V^{\mathbb{P}}$. Moreover, working in $V^{\mathbb{P}}$, let χ be the statement that G is a filter in \mathbb{P} which meets D . χ is upwards absolute, so $\phi(\mathbb{P}, \mathcal{D})$ will remain true in any forcing extension of $V^{\mathbb{P}}$. This proves the claim. \square

So by $\text{MP}_{\Gamma}(H(\omega_2))$ and the claim, $\phi(\mathbb{P}, \mathcal{D})$ is true (in V), for arbitrary CCC forcing notion \mathbb{P} of size ω_1 and for any family of dense sets \mathcal{D} where $|\mathcal{D}| = \omega_1$, so MA_{ω_1} is true. \square

Lemma 3.9. *Let Γ be a class of forcing notions containing CCC. Then it is Γ -forceable that MA_{ω_1} is false.*

Proof. To find a Γ -forcing extension of V in which MA_{ω_1} is false it suffices to find one that contains a Suslin tree, since MA_{ω_1} implies that there are none. But forcing to add a single Cohen real will introduce a Suslin tree (see [JECH], Theorem 28.12, or [TF], chapter III). This forcing notion is CCC, hence in Γ . \square

Proof of Theorem 3.7. Assume $V \models \Box \text{MP}_{\Gamma}(H(\omega_2))$, i.e. that $\text{MP}_{\Gamma}(H(\omega_2))$ is true and holds in any Γ -forcing extension. Thus, by Lemma 3.8, MA_{ω_1} is also true in any Γ -forcing extension. But by Lemma 3.9, one can always find a Γ -forcing extension in which MA_{ω_1} is false, a contradiction. So $\Box \text{MP}_{\Gamma}(H(\omega_2))$ is false. \square

Remark. This proof erases the possibility mentioned in the comment after Theorem 3.5 that MP_{CCC} directly implies $\text{MP}_{\text{CCC}}(H(\kappa))$ for any CCC-absolutely definable cardinal κ . This is because $\text{MP}_{\Gamma}(H(\omega_2))$ is false in the extension that adds one Cohen real. So “ MP_{CCC} implies $\text{MP}_{\text{CCC}}(H(\kappa))$ ” fails for $\kappa = \omega_1$.

Remark. Furthermore, this proof erases the possibility mentioned in the remark after the proof of Theorem 3.1 that MP_{CCC} directly implies $\text{MP}_{\text{CCC}}(\mathbb{R})$. If MP_{CCC} holds, then it is necessary. So if MP_{CCC} implied $\text{MP}_{\text{CCC}}(\mathbb{R})$ then it would imply $\Box \text{MP}_{\text{CCC}}(\mathbb{R})$, which implies ω_1 is inaccessible to reals, but MP_{CCC} is just equiconsistent with ZFC.

Corollary 3.10. *The following are all false:*

- (1) $\Box\text{MP}_{\text{CCC}}(H(\omega_2))$
- (2) $\Box\text{MP}_{\text{CCC}}(H(2^\omega))$
- (3) $\Box\text{MP}_{\text{CARD}}(H(\omega_2))$
- (4) $\Box\text{MP}_{\omega_1}(H(\omega_2))$
- (5) $\Box\text{MP}_{\text{PROPER}}(H(\omega_2))$
- (6) $\Box\text{MP}_{\text{SEMIPROPER}}(H(\omega_2))$

Proof. (1): See Theorem 3.7. (2): Under MP_{CCC} , the parameter set $H(2^\omega)$ includes $H(\omega_2)$, so $\Box\text{MP}_{\text{CCC}}(H(2^\omega))$ implies $\Box\text{MP}_{\text{CCC}}(H(\omega_2))$, which is false. (3)-(6): The forcing classes of these principles all contain CCC, so Theorem 3.7 applies. \square

A question these results still do not answer is, what is the consistency strength of $\Box\text{MP}_{\text{CCC}}(\mathbb{R})$?

3.3 ω_1 is inaccessible to reals under $\Box\text{MP}_{\text{CCC}}(\mathbb{R})$

Restricting parameters to the reals, we now ask about the consistency strength of $\Box\text{MP}_{\text{CCC}}(\mathbb{R})$. Hoping that this restriction on parameters will not lead to inconsistency, we can at least show that its consistency is strictly beyond that of ZFC. A cardinal δ is said to be **inaccessible to reals** if $L[r] \models$ “ δ is inaccessible” for any $r \in \mathbb{R}$, where $L[r]$ is the model of ZFC consisting of all sets constructible from r . In the case of $\delta = \omega_1$, the following standard characterization of this property is useful (see Problem 32.4 in [JECH], and Proposition 11.5 in [KAN].)

Lemma 3.11. *ω_1 is inaccessible to reals if and only if $\omega_1 \neq \omega_1^{L[z]}$ for any real z .*

Proof. \implies : If $\omega_1 = \omega_1^{L[z]}$ for some real z then $L[z] \models \omega_1 = \omega^+$, therefore $L[z] \models$ “ ω_1 is accessible” for the real z .

\impliedby : Let $\omega_1 \neq \omega_1^{L[z]}$ for any real z . Since ω_1 is regular in V , it must be regular in any submodel. Looking at the submodel $L[z]$, this means it will

suffice to show that ω_1 is a strong limit, i.e., that $L[z] \models 2^\kappa < \omega_1$ for every $\kappa < \omega_1$. Since $L[z]$ satisfies *GCH*, it is therefore sufficient to show that, in $L[z]$, ω_1 is not the successor of any cardinal κ . Working in $L[z]$, suppose towards contradiction that $\omega_1 = \kappa^+$. Then $\kappa < \omega_1$, so κ is a countable ordinal in V . So we can code the pair $\langle \kappa, z \rangle$ with a real w . We move to $L[w]$, where $L[z] \subseteq L[w]$ and, by definition, $L[w] \models |\kappa| = \omega$. So $\omega_1 = \omega_1^{L[w]}$ for the real w , a contradiction. \square

The next lemma constructs a CCC-absolute sequence, which is used in a subsequent theorem.

Lemma 3.12. *There is a family of functions $\{e_\alpha : \alpha \rightarrow \omega \mid \omega \leq \alpha < \omega_1\}$ with the property that, for all $\alpha < \omega_1$,*

(1) e_α is 1-1.

(2) For all β such that $\omega \leq \beta < \alpha$, $|\{\xi < \beta \mid e_\alpha(\xi) \neq e_\beta(\xi)\}| < \omega$
(any disagreement between e_α and e_β is finite).

Remark. Such a sequence of functions is called an **almost-coherent sequence**.

Proof. Fix a \subseteq^* -descending sequence $\{A_\alpha\}_{\alpha < \omega_1}$ of subsets of ω (that is, such that for all $\alpha < \beta < \omega_1$, $|A_\beta \setminus A_\alpha| < \omega$ and $|A_\alpha \setminus A_\beta| = \omega$). Using this sequence we will construct $\{e_\alpha\}$ inductively, by imposing a third condition at each stage:

(3) $\text{range}(e_\alpha) \cap A_\alpha = \emptyset$.

Basis step ($\alpha = \omega$): Let e_ω be any 1-1 function from ω into ω which avoids A_ω .

Induction step: Let $\delta < \omega_1$. We assume that e_α is defined and satisfies (1)–(3) for all $\omega \leq \alpha < \delta$.

First, suppose $\delta = \alpha + 1$, a successor. Define $e_{\alpha+1}(\xi) = e_\alpha(\xi)$ for all $\xi \in \alpha$, unless $e_\alpha(\xi) \in A_{\alpha+1}$. In the latter case, by (3) and the induction hypothesis, $e_\alpha(\xi) \notin A_\alpha$, that is, $e_\alpha(\xi) \in A_{\alpha+1} \setminus A_\alpha$, a finite set, so choose all such $e_{\alpha+1}(\xi)$ to have distinct values in $A_\alpha \setminus A_{\alpha+1}$, an infinite set. (1)–(3) will thus be preserved.

Next, suppose δ is a (countable) limit ordinal: enumerate the set of all ordinals below δ as an ω sequence. There then will be a monotonically

increasing subsequence of such ordinals, $\{\alpha_n\}_{n<\omega}$, unbounded in δ and beginning with α_0 , an arbitrary ordinal below δ . Our strategy will be to define a sequence of functions $\{e_\delta^n : \alpha_n \rightarrow \omega\}_{n<\omega}$ whose union will be our desired function e_δ .

Claim: We can define such a sequence to have the following properties, for all $n < \omega$:

- (i) $e_\delta^{n+1} \upharpoonright \alpha_n = e_\delta^n$. (This ensures that $e_\delta = \cup_{i<\omega} e_\delta^i$ is a function.)
- (ii) $e_\delta^n =^* e_{\alpha_n}$.
- (iii) $range(e_\delta^n) \cap A_\delta = \emptyset$.
- (iv) e_δ^n is 1-1.
- (v) $domain(e_\delta^n) = \alpha_n$.

(We write $f =^* g$, of two functions when they disagree in only finitely many places.)

From the claim we can establish the induction step of our original argument, and the lemma will be proved:

(1) Suppose e_δ is not 1-1. Then by cofinality of $\{\alpha_n\}$ in δ and (v), there must be $n < \omega$ such that e_δ^n is not 1-1, contradiction.

(2) For all $n < \omega$, $e_\delta \upharpoonright \alpha_n =_{(i)} e_\delta^n =_{(ii)}^* e_{\alpha_n}$, which is what (2) says in the case of $\alpha_n < \delta$. Moreover, for any ordinal $\beta < \delta$, there is $n < \omega$ such that $\beta < \alpha_n < \delta$, since $\{\alpha_n\}$ has been defined to be cofinal in δ . By the induction hypothesis, the disagreement of e_β with e_{α_n} is finite, and since this is also the state of affairs between e_{α_n} and e_δ , so also must it be between e_β and e_δ .

(3) $range(e_\delta) \cap A_\delta = range(\cup_{n<\omega} e_\delta^n) \cap A_\delta = \cup_{n<\omega} (range(e_\delta^n) \cap A_\delta) = \emptyset$.

Proof of claim (by induction over $n < \omega$):

Basis step: define $e_\delta^0 : \alpha_0 \rightarrow \omega$ by emulating the successor case in the inductive definition of e_δ : For $\xi \in \alpha_0$, let $e_\delta^0(\xi) = e_{\alpha_0}(\xi)$, unless $e_{\alpha_0}(\xi) \in A_\delta$. By (3), $e_{\alpha_0}(\xi) \notin A_{\alpha_0}$, so $e_{\alpha_0}(\xi) \in A_\delta \setminus A_{\alpha_0}$, a finite set. So let $e_\delta^0(\xi)$ take an unused value from $A_{\alpha_0} \setminus A_\delta$ (an infinite set) in this case. Thus (ii)–(v) are satisfied.

Induction step: For all $\xi \in \alpha_n$, define $e_\delta^{n+1}(\xi) = e_\delta^n(\xi)$ (this gives (i)). For $\xi \in [\alpha_n, \alpha_{n+1})$, proceed as in the basis step to establish (ii)–(v): Let $e_\delta^{n+1}(\xi) = e_{\alpha_{n+1}}(\xi)$, unless $e_{\alpha_{n+1}}(\xi) \in A_\delta$, and so forth. \square

Note that the essential properties of the almost-coherent sequence constructed are CCC-absolute: the cardinal ω_1 is preserved, and the definition is Δ_0 .

We now content ourselves to quote a theorem from [TF], after some introductory definitions. Suppose $a : [\omega_1]^2 \rightarrow \omega$ is such that if one defines $a_\beta : \omega_1 \rightarrow \omega$ by $a_\beta(\alpha) = a(\alpha, \beta)$, then the family $\{a_\beta \mid \beta < \omega_1\}$ is almost coherent. Set $T(a) = \{a_\beta \upharpoonright \alpha : \alpha \leq \beta < \omega_1\}$, the set of all initial segments of the functions a_β . Then $T(a)$ is an ω_1 -tree ordered by inclusion: the chains have length ω since initial segments of injective functions from ω_1 to ω can only be extended for a countable chain. And each node has finite branching, so the cardinality of each level is countable. Note that the almost coherent sequence of Lemma 3.12 can be represented by a function $e : [\omega_1]^2 \rightarrow \omega$.

Let \mathcal{C}_ω be the forcing notion consisting of finite partial functions from ω to ω . (This forcing notion is CCC). The union of a resulting \mathcal{C}_ω -generic filter is then a generic function from ω to ω , which we identify with the corresponding ‘‘Cohen’’ subset of ω , or Cohen real. If $c : \omega \rightarrow \omega$ and $e : [\omega_1]^2 \rightarrow \omega$, define $e_c = ce : [\omega_1]^2 \rightarrow \omega$.

Theorem 3.13 (Todorćević). *If c is a \mathcal{C}_ω -generic function from ω to ω and $e : [\omega_1]^2 \rightarrow \omega$ is almost coherent, then $T(e_c)$ is a Suslin tree.*

The next theorem is the main result of this section.

Theorem 3.14. $\square\text{MP}_{\text{CCC}}(\mathbb{R})$ *implies that ω_1 is inaccessible to reals.*

Proof. Assume $\square\text{MP}_{\text{CCC}}(\mathbb{R})$. By Lemma 3.11 it is sufficient to show $\omega_1 \neq \omega_1^{L[z]}$ for any real z . So suppose, towards contradiction, that $\omega_1 = \omega_1^{L[z]}$ for some $z \in \mathbb{R}$. By Lemma 3.12 $L[z]$ has an almost coherent family of functions $\{c_\alpha : \alpha \rightarrow \omega \mid \alpha < \omega_1\}$, which is definable in a way that is absolute to forcing extensions. And since $\omega_1^{L[z]} = \omega_1$, the family, in V , still maps all initial segments of ω_1 to ω . And $L[z]$ has the correct size of the family. Let $e = \{e_\alpha \mid \alpha < \omega_1\}$ be the $L[z]$ -least such family. The point here is that this defines e in any forcing extension. So, in the CCC-forcing extension that adds the Cohen real c , the tree $T = T(ce)$, as provided by Theorem 3.13, is definable from c and z . Moreover, the definition is absolute to any extension.

Now consider the statement (with real parameters) $\phi(z, c) =$ ‘‘The tree constructed in $L[z]$ with the real c , $T(ce)$, has an ω_1 -branch’’. This formula has z and c as parameters since the tree $T(ce)$ is definable from them. In $V[c]$, $\phi(z, c)$ is false, since $T(ce)$ is Suslin by Theorem 3.13. But by CCC-forcing

(using the tree itself) an ω_1 -branch appears, so $\phi(z, c)$ is CCC-necessary in this extension. By the principle $\text{MP}_{\text{CCC}}(\mathbb{R})$, $\phi(z, c)$ must be true in $V[c]$, a contradiction. \square

Chapter 4

MP_{COHEN} and variations

Let δ be an uncountable ordinal. Recall that $Add(\omega, \delta)$ is the forcing notion, consisting of finite partial functions from $\omega \times \delta = \delta$ to 2, that adds δ new reals. We define the class of forcing notions COHEN to be $\{Add(\omega, \theta) \mid \theta \text{ a cardinal}\}$. In this chapter we establish equiconsistency of forms of the principle MP_{COHEN} with ZFC. We will make use of the fact that, if $\theta < \delta$ are cardinals, then forcing with $Add(\omega, \theta)$ followed by forcing with $Add(\omega, \delta)$ is the same as forcing with $Add(\omega, \delta)$ alone. In fact, these forcing notions are absolute, since their conditions are all finite, so all the names are in the ground model and this two-step iteration is just product forcing. So $Add(\omega, \delta) * Add(\omega, \alpha) \sim Add(\omega, \alpha) * Add(\omega, \delta) \sim Add(\omega, \delta)$, where \sim denotes forcing-equivalence.

Theorem 4.1. *If there is a model of $ZFC + V_\delta \prec V$ then there is a model of $ZFC + MP_{COHEN}$.*

Proof. We find such a model by forcing with $Add(\omega, \delta)$. Let G be V -generic over $Add(\omega, \delta)$. Note that G has size δ . We will show that $V[G] \models MP_{COHEN}$. Suppose $V[G] \models \text{“}\phi \text{ is COHEN-forceably necessary”}$ for an arbitrarily chosen sentence ϕ in the language of ZFC. Now notice that ϕ is COHEN-forceably necessary in V as well as $V[G]$ — ϕ is forced necessary by $Add(\omega, \delta) * Add(\omega, \alpha) \sim Add(\omega, \delta + \alpha)$ for some α . Let α be the least α such that $V[G] \models Add(\omega, \alpha)$ forces that “ ϕ is COHEN-necessary”. We have $\alpha < \delta$, whence $|\delta + \alpha| = \delta$, because α is definable and hence in V_δ by elementarity. But $Add(\omega, \delta) * Add(\omega, \alpha) \sim Add(\omega, \delta)$ forces “ ϕ is COHEN-necessary”. So $V[G] \models \text{“}\phi \text{ is COHEN-necessary”}$. \square

Theorem 4.2. *If there is a model of $\text{ZFC} + V_\delta \prec V + \text{cof}(\delta) > \omega$ then there is a model of $\text{ZFC} + \text{MP}_{\text{COHEN}}(\mathbb{R})$.*

Proof. We use the same model as in Theorem 4.1, by forcing with $\text{Add}(\omega, \delta)$. Let G be V -generic over $\text{Add}(\omega, \delta)$. We will show that $V[G] \models \text{MP}_{\text{COHEN}}(\mathbb{R})$. Suppose $V[G] \models \text{“}\phi(r) \text{ is COHEN-forceably necessary and } r \in \mathbb{R}\text{”}$. Since $\text{cof}(\delta) > \omega$, r is added to $V[G]$ at some “stage”, i.e., by $\text{Add}(\omega, \theta)$ for $\theta < \delta$ (if not, and unboundedly many $\theta < \delta$ are needed to decide r , then δ has cofinality ω , contradiction). So $r \in V_\delta[r] \subseteq V[G \restriction \theta]$ for $\theta < \delta$, where $G \restriction \theta = G \cap \text{Add}(\omega, \theta)$. Since $V[G] \models \text{“}\phi(r) \text{ is COHEN-forceably necessary and } r \in \mathbb{R}\text{”}$, let α be the least α such that $V[G] \models \text{Add}(\omega, \alpha)$ forces that “ $\phi(r)$ is COHEN-necessary”. We have $\alpha < \delta$, because it is definable from r , θ , and $G \restriction \theta$ and hence in V_δ by elementarity. According to V , $\text{Add}(\omega, \delta) * \text{Add}(\omega, \alpha) \Vdash \cong \text{Add}(\omega, \delta)$ forces “ $\phi(r)$ is COHEN-necessary”. So $V[G] \models \text{“}\phi(r) \text{ is COHEN-necessary”}$. \square

Remark. The use of $V_\delta \prec V$ in this proof was not to circumvent the undefinability of truth, as in earlier proofs, but rather to obtain the forcing of the COHEN-necessity of $\phi(x)$ by $\text{Add}(\omega, \theta)$ where $\theta < \delta$. This raises the possibility that these arguments share a common thread involving reflection which is deeper than what has been presented here.

Theorem 4.3. *If there is a model of $\text{ZFC} + V_\delta \prec V + \text{cof}(\delta) > \omega$ then there is a model of $\text{ZFC} + \Box \text{MP}_{\text{COHEN}}(\mathbb{R})$.*

Proof. Take the model $V[G]$ from Theorem 4.2, with G V -generic over $\text{Add}(\omega, \delta)$, where $V_\delta \prec V$. In $V[G]$, force with $\text{Add}(\omega, \beta)$ for some β . Let $H \subseteq \text{Add}(\omega, \beta)$ be $V[G]$ -generic (so new reals are added to $V[G]$). Consider $V[G][H]$, and fix some real x in $V[G][H]$. Then x is decided by a countable subset of $\text{Add}(\omega, \beta)$; call it \mathbb{P}_0 . So x is in $V[G][H_0]$, where $H_0 = H \cap \mathbb{P}_0 \subseteq \mathbb{P}_0$. Since \mathbb{P}_0 adds the real x and is countable, $\mathbb{P}_0 \cong \text{Add}(\omega, 1)$. This gives $V[G][H_0] = V[G * H_0]$, where $G * H_0$ is V -generic over $\text{Add}(\omega, \delta) * \text{Add}(\omega, 1) \cong \text{Add}(\omega, \delta)$. So $V[G * H_0]$ satisfies the conditions of Theorem 4.2, so $V[G * H_0] \models \text{MP}_{\text{COHEN}}(\mathbb{R}^{V[G * H_0]})$. But x is in $\mathbb{R}^{V[G * H_0]}$, so adding it did not invalidate $\text{MP}_{\text{COHEN}}(\mathbb{R})$. So $V[G] \models \Box \text{MP}_{\text{COHEN}}(\mathbb{R})$. \square

Corollary 4.4. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC})$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COHEN}})$

(3) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COHEN}}(\mathbb{R}))$

(4) $\text{Con}(\text{ZFC} + \Box \text{MP}_{\text{COHEN}}(\mathbb{R}))$.

Proof. (4) \implies (3) \implies (2) \implies (1): Obvious.

(1) \implies (4): By Lemma 1.21, we know that it is equiconsistent with ZFC that the ground model V satisfies $V_\delta \prec V$ for some cardinal δ , with $\text{cof}(\delta) > \omega$. And by Theorem 4.3, we know, given such a model of $\text{ZFC} + V_\delta \prec V + \text{cof}(\delta) > \omega$, that there is a forcing extension model of $\text{ZFC} + \Box \text{MP}_{\text{COHEN}}(\mathbb{R})$. \square

These equiconsistency results immediately raise questions as to whether any of them are, in fact, equivalences.

Question 3. Is MP_{COHEN} equivalent to $\text{MP}_{\text{COHEN}}(\mathbb{R})$?

Question 4. Is $\text{MP}_{\text{COHEN}}(\mathbb{R})$ equivalent to $\Box \text{MP}_{\text{COHEN}}(\mathbb{R})$?

Theorem 4.5. *If there is a model of $\text{ZFC} + V_\delta \prec V$, then there is a COHEN-forcing extension which models $\text{ZFC} + \text{MP}_{\text{COHEN}}(H(\text{cof}(\delta)))$.*

Proof. The proof is almost identical to Theorem 4.2. Again, we force with $\text{Add}(\omega, \delta)$. The single difference is that the parameter of the formula ϕ is taken from $H(\text{cof}(\delta))$. Since it still has size less than $\text{cof}(\delta)$, the parameter appears from forcing with $\text{Add}(\omega, \theta)$ for some $\theta < \delta$, and the proof proceeds in the same way. \square

Remark. Since the final model is $V[G]$ where G is V -generic over $\text{Add}(\omega, \delta)$, it is clear that $2^\omega = \delta$ in $V[G]$. As long as δ is larger than the continuum in the ground model, the model created is a model of $\text{MP}_{\text{COHEN}}(H(\text{cof}(2^\omega)))$.

Corollary 4.6. *The following are equivalent:*

(1) $\text{Con}(\text{ZFC})$

(2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COHEN}}(H(\text{cof}(2^\omega))))$

(3) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COHEN}}(H(\omega_{17})) + 2^\omega = \omega_{17})$

(4) $\text{Con}(\text{ZFC} + \Box \text{MP}_{\text{COHEN}}(H(\text{cof}(2^\omega))))$.

(5) $\text{Con}(\text{ZFC} + \Box \text{MP}_{\text{COHEN}}(H(\omega_{17})) + 2^\omega = \omega_{17})$.

Proof. Lemma 1.21 can be used in the proof of Theorem 4.2 to allow the choice of δ to have cofinality larger than any particular definable cardinal. And by modifying the proof of Theorem 4.3 in the obvious way, we get the necessary versions of these corollaries. \square

Lemma 4.7. *If $\text{MP}_{\text{COHEN}}(2^\omega)$ then $2^\omega = 2^\kappa$ for all infinite $\kappa < 2^\omega$.*

Proof. Let $V \models \text{MP}_{\text{COHEN}}(2^\omega)$. Suppose $\omega < \kappa < 2^\omega$. Let G be V -generic over $\mathbb{P} = \text{Add}(\omega, \lambda)$ where $\lambda = (2^\kappa)^V$. We compute the size of 2^κ in the extension $V[G]$, i.e., how many subsets of κ have been added by forcing with \mathbb{P} . We can bound $(2^\kappa)^{V[G]}$ above by counting the nice \mathbb{P} -names for subsets of κ . \mathbb{P} is in CCC, and $|\mathbb{P}| = \lambda = (2^\kappa)^V$, so there are λ^ω antichains in \mathbb{P} . So there are $(\lambda^\omega)^\kappa = (2^\kappa)^{\omega \cdot \kappa} = 2^\kappa$ nice \mathbb{P} -names in V . So the number of subsets of κ does not increase when forcing with \mathbb{P} , i.e., $(2^\kappa)^{V[G]} = (2^\kappa)^V$. But $\mathbb{P} = \text{Add}(\omega, \lambda)$ has changed 2^ω to $\lambda = (2^\kappa)^V = (2^\kappa)^{V[G]}$. So $V[G] \models 2^\omega = 2^\kappa$. Moreover, further COHEN-forcing cannot destroy this statement: Since $\omega < \kappa$, $2^\omega \leq 2^\kappa$ is absolute. And $2^\omega < 2^\kappa$ is impossible in any further COHEN-forcing extension, by a similar nice names argument: Let $\mathbb{Q} = \text{Add}(\omega, \mu)$, where $V[G] \models \mu > 2^\omega$, and let H be $V[G]$ -generic over \mathbb{Q} . Working in $V[G]$, the number of nice \mathbb{Q} -names of subsets of κ is $(\mu^\omega)^\kappa = \mu^\kappa = \mu$ (since $\omega < \kappa < \mu$). So $V[G][H] \models 2^\kappa = \mu = 2^\omega$. This shows that $2^\omega = 2^\kappa$ is COHEN-forceably necessary. Defining $\phi(\kappa)$ to be “ $2^\kappa = 2^\omega$ ”, κ is a parameter allowed by $\text{MP}_{\text{COHEN}}(2^\omega)$. Hence, by applying $\text{MP}_{\text{COHEN}}(2^\omega)$, $2^\kappa = 2^\omega$ is true. \square

Lemma 4.8. *If there is a model of $\text{ZFC} + \text{MP}_{\text{COHEN}}(H(2^\omega))$, then it has an inner model of $\text{ZFC} + V_\delta \prec V + “\delta \text{ is inaccessible}”$.*

Proof. We first show $\text{MP}_{\text{COHEN}}(H(2^\omega))$ implies that 2^ω is weakly inaccessible. By Lemma 4.7 2^ω is regular (for all $\kappa < 2^\omega$, $\text{cof}(2^\omega) = \text{cof}(2^\kappa) > \kappa$). And 2^ω cannot be a successor cardinal: for any cardinal κ , $\text{Add}(\omega, \kappa^{++})$ forces that it is COHEN-necessary that $2^\omega > \kappa^+$, so that must be true. So 2^ω is a regular limit cardinal. Since it is weakly inaccessible, L models that 2^ω is (strongly) inaccessible.

Next we show that, if $\delta = 2^\omega$, $L_\delta \prec L$. Suppose $L \models \exists y \psi(a, y)$ for the formula $\exists y \psi(a, y)$ with parameter a . Let α be the least cardinal such that $\exists y \in L_\alpha$ such that $\psi(a, y)$. Consider $\phi(a) = “\text{the least } \alpha \text{ such that there is a } y \text{ in } L_\alpha \text{ with } \psi(a, y)^L \text{ is less than } 2^\omega”$. This is expressed using the parameter a in $L_\delta \subseteq H(\delta)$. Since this is COHEN-forceably necessary, it is true. So there is a y in $L_\alpha \subseteq L_\delta$ such that $\psi(a, y)$. So by the Tarski-Vaught criterion, $L_\delta \prec L$, and L is the desired inner model. \square

Lemma 4.9. *If there is a model of $\text{ZFC} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”}$ then there is a model of $\text{ZFC} + \text{MP}_{\text{COHEN}}(H(2^\omega))$.*

Proof. Suppose $V \models \text{ZFC} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”}$. Let G be V -generic over $\mathbb{P} = \text{Add}(\omega, \delta)$. We claim that $V[G] \models \text{MP}_{\text{COHEN}}(H(\delta))$. Suppose $V[G] \models \text{“}\phi(x) \text{ is COHEN-forceably necessary and } x \text{ is in } H(\delta)\text{”}$. It will suffice to show $V[G] \models \text{“}\phi(x) \text{ is COHEN-necessary”}$. Since x has size less than δ , which is regular, it must be added by $\text{Add}(\omega, \theta)$ for some $\theta < \delta$.

Just as in the proof of Theorem 4.2, let α be the least α such that $V[G] \models \text{“}\text{Add}(\omega, \alpha) \text{ forces that } \phi(x) \text{ is COHEN-necessary”}$. We have $\alpha < \delta$, because it is definable from x , θ , and $G \restriction \theta$ and hence in V_δ by elementarity. According to V , $\text{Add}(\omega, \delta) * \text{Add}(\omega, \alpha) \cong \text{Add}(\omega, \delta) \Vdash \text{“}\phi(x) \text{ is COHEN-necessary”}$. So $V[G] \models \text{“}\phi(x) \text{ is COHEN-necessary”}$. \square

Theorem 4.10. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COHEN}}(H(2^\omega)))$
- (2) $\text{Con}(\text{ZFC} + V_\delta \prec V + \text{“}\delta \text{ is an inaccessible cardinal”})$
- (3) $\text{Con}(\text{ZFC} + \text{“ORD is Mahlo”})$.

Proof. The equivalence of (1) with (2) is from Lemma 4.7 and Lemma 4.8. The proof of the equivalence of (2) with (3) is in [HAM1]. \square

So far a strong analogy seems to exist between the results concerning consistency strengths of maximality principles for the forcing class COHEN and the class CCC. We pursue this correspondence further into the boxed (necessary) versions of these principles.

Theorem 4.11. $\square_{\text{MP}_{\text{COHEN}}}(H(2^\omega))$ is false.

Proof. By Lemma 4.7, $\text{MP}_{\text{COHEN}}(H(2^\omega))$ implies that 2^ω is weakly inaccessible. So $\square_{\text{MP}_{\text{COHEN}}}(H(2^\omega))$ implies that 2^ω is weakly inaccessible in every COHEN-forcing extension. But there are COHEN-forcing extensions that add any desired cardinality of reals, including successor cardinalities, which are therefore not weakly inaccessible. \square

Chapter 5

MP_{COLL} and variations

Let δ be an uncountable ordinal. Recall that $\text{Col}(\omega, \theta)$ is the forcing notion, consisting of finite partial injective functions from θ to ω , that collapses the cardinal δ to ω . We will use the notation $\text{Col}(\omega, < \delta)$ to represent the Lévy collapse of δ to ω_1 , for limit cardinal δ . We define the class of forcing notions COLL to be $\{\text{Col}(\omega, \theta) \mid \theta \text{ a cardinal}\} \cup \{\text{Col}(\omega, < \theta) \mid \theta \text{ a limit cardinal}\}$, and we include, for technical reasons, all forcing notions that are forcing-equivalent to some $\text{Col}(\omega, \theta)$ or $\text{Col}(\omega, < \theta)$. Two forcing notions are forcing-equivalent if they produce the same forcing extensions, i.e., they have isomorphic regular open Boolean Algebras. This does not alter the meaning of MP_{COLL} or any variation thereof, since saying “ M is a COLL -forcing extension” means the same thing in either interpretation of COLL . One sees it is natural to include the Lévy collapse with forcing notions that collapse any θ to ω by recognizing that $\text{Col}(\omega, \theta)$ is equivalent to $\text{Col}(\omega, < \theta^+)$. That is, $\text{Col}(\omega, \theta)$ collapses the entire interval $(\omega, \theta]$, while $\text{Col}(\omega, < \theta)$ collapses the interval (ω, θ) . So there are natural closure properties for this class.

In this chapter we establish equiconsistency of forms of the principle MP_{COLL} with ZFC. We will make use of the fact that, if $\theta < \delta$ are cardinals, with δ a limit cardinal, then forcing with $\text{Col}(\omega, < \delta)$ followed by forcing with $\text{Col}(\omega, \theta)$ is the same as forcing with $\text{Col}(\omega, < \delta)$ alone. This chapter retraces most of the same kinds of arguments as found in the previous chapter.

Recall the definition of the absorption relation between two classes of forcing notions: Γ_2 **absorbs** Γ_1 if it is true and Γ_1 -necessary that Γ_2 is contained in Γ_1 and for any \mathbb{P} in Γ_1 there exists \mathbb{Q} such that $V^{\mathbb{P}} \models “\dot{\mathbb{Q}} \text{ is in } \Gamma_2”$ and “ $\mathbb{P} * \dot{\mathbb{Q}}$ is in Γ_2 ”. There is a well-known fact that can be cast in this language.

Lemma 5.1. *The class COLL absorbs all forcing. That is, if \mathbb{P} is any forcing notion, then there is \mathbb{Q} in COLL (according to $V^{\mathbb{P}}$) such that $\mathbb{P} * \mathbb{Q}$ is in COLL .*

To prove this, one makes use of a lemma that Solovay used in his famous Lebesgue measurability result. It can be found as Proposition 10.20 in [KAN].

Lemma 5.2 (Solovay). *Let \mathbb{P} be a separative forcing notion of size $\leq \alpha$ which collapses α to ω . Then there is a dense subset of $\text{Col}(\omega, \alpha)$ that densely embeds into \mathbb{P} .*

Proof of Lemma 5.1. Let \mathbb{P} be any forcing notion. Pick cardinal α to be at least $|\mathbb{P}|$, and let $\mathbb{Q} = \text{Col}(\omega, \alpha)$. Then $|\mathbb{P} * \mathbb{Q}| = \alpha$, and $\mathbb{P} * \mathbb{Q}$ collapses α to ω . So by Lemma 5.2 there is a dense embedding from a dense subset of $\mathbb{Q} = \text{Col}(\omega, \alpha)$ into $\mathbb{P} * \mathbb{Q}$. So, as forcing notions, $\mathbb{P} * \mathbb{Q}$ is equivalent to $\text{Col}(\omega, \alpha)$, which is in COLL . So $\mathbb{P} * \mathbb{Q}$ itself is in COLL . \square

Corollary 5.3. MP_{COLL} implies MP .

Proof. By Lemma 5.1 and Corollary 2.7. \square

Corollary 5.4. $\text{MP}_{\text{COLL}}(\mathbb{R})$ implies $\text{MP}(\mathbb{R})$.

Proof. By Lemma 5.1 and Corollary 2.8. \square

Theorem 5.5. *If there is a model of $\text{ZFC} + V_\delta \prec V$ then there is a model of $\text{ZFC} + \text{MP}_{\text{COLL}}$.*

Proof. We find such a model by forcing with $\text{Col}(\omega, < \delta)$. Let G be V -generic over $\text{Col}(\omega, < \delta)$. Note that G has size δ . We will show that $V[G] \models \text{MP}_{\text{COLL}}$. Suppose $V[G] \models \text{“}\phi \text{ is COLL-forceably necessary”}$ for an arbitrarily chosen sentence ϕ in the language of ZFC . So $V[G] \models \text{“there is some member of COLL that forces that } \phi \text{ is COLL-necessary”}$. So some finite condition in $\text{Col}(\omega, < \delta)$ decides that some $\text{Col}(\omega, \lambda)$ forces that ϕ is COLL-necessary. So there is $\theta < \delta$ such that $V[G \restriction \theta] \models \text{“}\phi \text{ is COLL-forceably necessary”}$. Now notice that ϕ is COLL-forceably necessary in V as well as $V[G]$ — ϕ is forced necessary by $\text{Col}(\omega, < \delta) * \text{Col}(\omega, \alpha)$ for some α . Let α be the least α such that $V[G \restriction \theta] \models \text{Col}(\omega, \alpha)$ forces that “ ϕ is COLL-necessary”. We have $\alpha < \delta$ because α is definable from $G \restriction \theta$ and hence in $V_\delta[G \restriction \theta]$ (and hence in $V_\delta[G]$) by elementarity. But $\text{Col}(\omega, < \delta) * \text{Col}(\omega, \alpha) \cong \text{Col}(\omega, < \delta)$ forces “ ϕ is COLL-necessary”. So $V[G] \models \text{“}\phi \text{ is COLL-necessary”}$. \square

Corollary 5.6. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC})$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COLL}})$

Theorem 5.7. *If there is a model of $\text{ZFC} + V_\delta \prec V + “\delta \text{ is inaccessible}”$ then there is a model of $\text{ZFC} + \text{MP}_{\text{COLL}}(\mathbb{R})$.*

Proof. Let V satisfy $\text{ZFC} + V_\delta \prec V + “\delta \text{ is inaccessible}”$. Now force with $\text{Col}(\omega, < \delta)$. Let G be V -generic over $\text{Col}(\omega, < \delta)$. We will show that $V[G] \models \text{MP}_{\text{COLL}}(\mathbb{R})$. Suppose $V[G] \models “\phi(r) \text{ is COLL-forceably necessary and } r \in \mathbb{R}”$. Since δ is inaccessible, $\text{Col}(\omega, < \delta)$ has the δ -cc, so r is added to $V[G]$ “at some earlier stage”, i.e., by $\text{Col}(\omega, \theta)$ for $\theta < \delta$. So $r \in V_\delta[r] \subseteq V_\delta[G \restriction \theta]$ for $\theta < \delta$, where $G \restriction \theta = G \cap \text{Col}(\omega, \theta)$. Since $|G \restriction \theta| < \delta$, forcing with $G \restriction \theta$ is “small” forcing, and so, by Lemma 1.18, $V_\delta[G \restriction \theta] \prec V[G \restriction \theta]$. Working in $V[G \restriction \theta]$, since $V[G \restriction \theta] \models “\phi(r) \text{ is COLL-forceably necessary and } r \in \mathbb{R}”$, let α be the least α such that $V[G \restriction \theta] \models \text{Col}(\omega, \alpha)$ forces that “ $\phi(r)$ is COLL-necessary”. We have $\alpha < \delta$, because it is definable from r , θ , and $G \restriction \theta$ and hence in $V_\delta[G \restriction \theta]$ by elementarity. According to V , $\text{Col}(\omega, < \delta) * \text{Col}(\omega, \alpha) \cong \text{Col}(\omega, < \delta)$ forces “ $\phi(r)$ is COLL-necessary”. So $V[G] \models “\phi(r) \text{ is COLL-necessary}”$. \square

Remark. This is another alternate use of $V_\delta \prec V$, not to circumvent the undefinability of truth, but to obtain the forcing of the COLL-necessity of $\phi(x)$ by $\text{Col}(\omega, \theta)$ where $\theta < \delta$.

Theorem 5.8. *If there is a model of $\text{ZFC} + \text{MP}_{\text{COLL}}(\mathbb{R})$ then it has an inner model of $\text{ZFC} + V_\delta \prec V + “\delta \text{ is inaccessible}”$. Specifically, ω_1 is inaccessible in L and $L_{\omega_1} \prec L$.*

Proof. This proof just mimics the proof in [HAM1] of the analogous result for $\text{MP}(\mathbb{R})$. First, we see that $\text{MP}_{\text{COLL}}(\mathbb{R})$ implies that ω_1 is inaccessible to reals: For any real x , let $\phi(x)$ be “the ω_1 of $L[x]$ is countable”, which implies $\omega_1^{L[x]} < \omega_1$, which implies $\delta = \omega_1$ is inaccessible in L . But this is clearly COLL-forceably necessary, by $\text{Col}(\omega, \omega_1)$, so it is true. Further, we get $L_{\omega_1} \prec L$, by the Tarski-Vaught argument. Suppose $L \models \exists y \psi(a, y)$ for a in L_{ω_1} . Let α be the least such that there is a y in L_α such that $\psi(a, y)$. Consider formula ϕ which is “the least α such that there is a y in L_α such that $\psi(a, y)$ is countable”. This is COLL-forceably necessary, using $\text{Col}(\omega, \alpha)$, hence true by $\text{MP}_{\text{COLL}}(\mathbb{R})$. \square

Corollary 5.9. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC} + \text{MP}_{\text{COLL}}(\mathbb{R}))$
- (2) $\text{Con}(\text{ZFC} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”})$
- (3) $\text{Con}(\text{ZFC} + \text{“ORD is Mahlo”})$.

Proof. (2) \iff (3): Proven in the analogous result for $\text{MP}(\mathbb{R})$ in [HAM1].

(2) \implies (1): By Theorem 5.7.

(1) \implies (2): By Theorem 5.8. □

Chapter 6

Large cardinals and indestructibility

6.1 $\text{MP}_{\Gamma(\kappa)}$

Certain classes of forcing notions are defined from a particular cardinal κ . Examples are κ -cc, κ -closed, $< \kappa$ -directed closed, and so on. In this chapter we'll explore maximality principles based on such classes, striving to obtain equiconsistency results by applying the same tools used earlier. Such classes will be collectively denoted by $\Gamma(\kappa)$, to show the role of κ in the definition of the class. The related maximality principles will then be written $\text{MP}_{\Gamma(\kappa)}$.

Let the formula $\sigma(x)$ express some cardinal property. Then, if there is a class Γ of forcing notions under which $\sigma(\kappa)$ is Γ -necessary, κ is said to be **indestructible** under forcing by Γ , or **Γ -indestructible**. In modal notation, this is expressed by $\Box_{\Gamma}\sigma(\kappa)$. The creation via forcing of a model of ZFC witnessing this indestructibility is called a **preparation**. A typical indestructibility result says that it is forceable that $\sigma(\kappa)$ is Γ -necessary (in modal notation, this is expressed as $\Diamond\Box_{\Gamma}\sigma(\kappa)$). Note that \Diamond has no subscript, indicating that the preparation itself need not be in the class Γ .

The literature contains a growing body of indestructibility results, beginning with Laver's preparation of a model of the indestructibility of a supercompact cardinal κ under $< \kappa$ -directed-closed forcing. Such a κ has come to be called **Laver-indestructible**. Due to its familiarity, it may be easiest to consider $\Gamma(\kappa)$ as being the class of $< \kappa$ -directed closed forcing notions (abbreviated $< \kappa$ -dc) and $\sigma(\kappa)$ as saying that κ is supercompact.

Laver's well-known result will then be expressed as $\Diamond \Box_{\Gamma(\kappa)} \sigma(\kappa)$. However, various other interpretations will also be possible. In fact, we first prove an equiconsistency result for MP_Γ in schematic form that will have, as instances, corollaries corresponding to the various indestructibility results known so far, as well as those yet to be discovered. In each of these cases the class Γ of forcing notions inevitably depends on the parameter κ , so this will be built into the scheme. So let $\Gamma(\kappa)$ denote a class of forcing notions that is defined in terms of the parameter κ , a cardinal. If κ is clear from the context of a statement, we may write Γ in place of $\Gamma(\kappa)$, leaving κ as an implicit parameter.

To prove this theorem, we first give a special case of Lemma 1.16.

Lemma 6.1. *Let Γ be a class of forcing notions, parametrized by κ . Let σ be any definable unary predicate. If there is a model M of $\text{ZFC} + \Box_\Gamma \sigma(\kappa)$ then there is a model of $\text{ZFC} + \Box_\Gamma \sigma(\kappa) + V_\delta \prec V$.*

Proof. Use Lemma 1.16, where theory T is $\text{ZFC} + \Box_\Gamma \sigma(\kappa)$. □

Remark. Again we are working, not merely in the language or model of ZFC , but in an expansion which has the constant κ . So if it is consistent that κ is indestructible by forcing with Γ , then it is also consistent that this holds and $V_\delta \prec V$ as well. One can assume that $\kappa < \delta$. This follows from the proof of Lemma 1.21. In fact, if $\kappa \geq \delta$, one could find a suitable $\kappa' < \delta$ by elementarity.

Remark. We can use the name κ as a constant in the language of both of the above equiconsistent theories to denote the same object in both. This is because, due to Lemma 1.17, a model of $V_\delta \prec V$ can be found by taking an elementary extension of any model of the theory on the left to obtain a model of the theory on the right. This is analogous with large cardinal results in which a forcing extension can provide new desired properties (such as indestructibility) for κ as interpreted by both the ground model and the extension. In the language of modal logic, κ becomes a rigid identifier in this context of different models connected by elementarity. This idea was described earlier, but only in the context of ground models as elementary substructures of forcing extensions.

Remark. Note that if Γ is S4 then $\Box_\Gamma \sigma(\kappa)$ directly implies $\sigma(\kappa)$, since the class Γ includes trivial forcing.

The next theorem draws on the argument in the proofs of Lemma 2.6 and Theorem 5.1 in [HAM1].

Theorem 6.2. *Let Γ be a class of forcing notions, parametrized by κ , which is Γ -necessarily closed under iterations of length ω using appropriate support, and which Γ -necessarily contains the trivial forcing. Further, let σ be a unary predicate. Then*

$\text{Con}(\text{ZFC} + \square_\Gamma \sigma(\kappa))$ if and only if $\text{Con}(\text{ZFC} + \square_\Gamma \sigma(\kappa) + \text{MP}_\Gamma)$.

Remark. Thus, if we make $\sigma(\kappa)$ indestructible by forcing in Γ , we have, at the same time, a model of MP_Γ .

Proof. The implication to the left is trivial. In the other direction, suppose $V \models \text{ZFC} + \square_\Gamma \sigma(\kappa)$. By Lemma 6.1 we can also assume $V \models V_\delta \prec V$. Let $\{\phi_n\}_{n \in \omega}$ enumerate all sentences in the language of ZFC. Now recursively define an ω -stage forcing iteration \mathbb{P} such that at each stage $n \in \omega$, $\mathbb{P}_n \in V_\delta$. Specifically, define $\mathbb{P}_0 = \{\emptyset\}$, the notion of trivial forcing. Next suppose \mathbb{P}_n has been defined, and in the model $V_\delta^{\mathbb{P}_n}$ define a \mathbb{P}_n -name for a forcing notion, \mathbb{Q}_n , as follows. If ϕ_n is Γ -forceably necessary then choose $\dot{\mathbb{Q}}_n$ to be the \mathbb{P}_n -name of a notion of forcing in $\Gamma^{V_\delta^{\mathbb{P}_n}}$ that forces $\square_\Gamma \phi_n$ ¹. Otherwise let $\dot{\mathbb{Q}}_n$ be the name for trivial forcing. Let $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_n$. Finally, let \mathbb{P} be the ω -iteration of $\{\mathbb{P}_n\}_{n \in \omega}$. Use appropriate support, as required by the closure conditions of the family Γ .

Let $G \subset \mathbb{P}$ be V -generic. I now claim that $V[G] \models \text{MP}_\Gamma$. To see this, let ϕ be a sentence in the language of ZFC which is Γ -forceably necessary in $V[G]$. I will show that $V[G] \models \square_\Gamma \phi$. First, ϕ must be ϕ_n for some $n \in \omega$. Factor $\mathbb{P} = \mathbb{P}_n * \mathbb{P}_{\text{TAIL}}$. This gives $V[G] = V[G_n][G_{\text{TAIL}}]$, taking respective generic filters. Since $V[G]$ is a Γ -forcing extension of $V[G_n]$, ϕ is Γ -forceably necessary in $V[G_n]$. And since $V_\delta \prec V$ and $\mathbb{P}_n \in V_\delta$, $V_\delta[G_n] \prec V[G_n]$ by Lemma 1.18. By elementarity, ϕ is Γ -forceably necessary in $V_\delta[G_n]$. But the iterated forcing \mathbb{P} was defined so that $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_n$, where \mathbb{Q}_n forces $\square_\Gamma \phi$ in $V_\delta^{\mathbb{P}_n}$. So take V -generic $G_{n+1} \subset \mathbb{P}_{n+1}$. We have $V_\delta[G_{n+1}] \models \square_\Gamma \phi$. So by elementarity again, $V[G_{n+1}] \models \square_\Gamma \phi$. But $V[G]$ is a Γ -forcing extension of $V[G_{n+1}]$, so $V[G] \models \square_\Gamma \phi$.

Finally, since \mathbb{P}_ω is in Γ , $\square_\Gamma \sigma(\kappa)$ still holds. \square

One doesn't need to limit applications of Theorem 6.2 to large cardinals.

¹This is exactly the point at which we need to be working in V_δ rather than V . Without a truth predicate—needed for a definition of the forcing relation—there is no formula applicable to uniformly express that ϕ_n is forceably necessary over V , for all n .

Theorem 6.3. *If there is a model of ZFC in which κ is any uncountable cardinal, and $\Gamma(\kappa)$ is a class of forcing notions preserving the cardinality of κ , defined using κ , which is $\Gamma(\kappa)$ -necessarily closed under iteration of length ω with appropriate support and which contains trivial forcing then there is a model of $\text{ZFC} + \text{MP}_{\Gamma(\kappa)}$.*

Proof. Let $\sigma(\kappa) = “\kappa \text{ is an uncountable cardinal}”$. So for κ , $\Box_{\Gamma(\kappa)}\sigma(\kappa)$. Now apply Theorem 6.2. \square

Let $<\kappa\text{-}dc$ denote the class of $<\kappa$ -directed closed notions of forcing.

Corollary 6.4. *The following are equivalent:*

- (1) $\text{Con}(\text{ZFC} + “\kappa \text{ is an uncountable cardinal}”)$
- (2) $\text{Con}(\text{ZFC} + \text{MP}_{<\kappa\text{-}dc} + “\kappa \text{ is an uncountable cardinal}”)$
- (3) $\text{Con}(\text{ZFC} + \text{MP}_{\kappa\text{-}closed} + “\kappa \text{ is an uncountable cardinal}”)$
- (4) $\text{Con}(\text{ZFC} + \text{MP}_{\kappa\text{-}cc} + “\kappa \text{ is an uncountable cardinal}”)$.

From existing indestructibility results arise various equiconsistency theorems, by using the following corollary of Theorem 6.2.

Corollary 6.5. *Let Γ be a class of forcing notions, parametrized by κ , which is Γ -necessarily closed under iterations of length ω using appropriate support, and which contains the trivial forcing. If there is a model of ZFC in which κ is a cardinal and $\sigma(\kappa)$ is a property of κ that can provably be made indestructible under Γ -forcing, then there is a model of $\text{ZFC} + \Box_{\Gamma}\sigma(\kappa) + \text{MP}_{\Gamma}$.*

Proof. In the forward direction, let $M_0 \models \text{ZFC} + \sigma(\kappa)$. By the hypothesis, there is also $M_1 \models \text{ZFC} + \Box_{\Gamma}\sigma(\kappa)$ holds. But then, by Theorem 6.2, there is $M \models \text{ZFC} + \Box_{\Gamma}\sigma(\kappa) + \text{MP}_{\Gamma}$. In the reverse direction, if we are given $M \models \text{ZFC} + \Box_{\Gamma}\sigma(\kappa) + \text{MP}_{\Gamma}$, then $M \models \text{ZFC} + \sigma(\kappa)$ since $\Box_{\Gamma}\sigma(\kappa)$ directly implies $\sigma(\kappa)$. \square

Corollary 6.6. *$\text{Con}(\text{ZFC} + “\text{There is a supercompact cardinal } \kappa”) \text{ if and only if } \text{Con}(\text{ZFC} + “\text{There is a supercompact cardinal } \kappa, \text{ indestructible by } <\kappa\text{-directed-closed forcing, such that } \text{MP}_{<\kappa\text{-}dc} \text{ holds}”).$*

Proof. By Laver’s indestructibility result in [LAV], Corollary 6.5, and the fact that the class of $<\kappa$ -directed closed forcing notions is closed under ω iterations using countable support. \square

A further result on indestructibility, in [GS], gives a preparation that renders a strong cardinal indestructible under a class of forcing notions, those that are κ^+ -weakly closed satisfying the Prikry condition. Herein I will use a weakening of this result, that strong cardinals can be made indestructible under $\leq \kappa$ -strategically closed forcing. We will call such a κ **Gitik-Shelah-indestructible**.

Corollary 6.7. *Con(ZFC + “There is a strong cardinal κ ”) \iff Con(ZFC + “There is a strong cardinal κ , Gitik-Shelah-indestructible, such that $\text{MP}_{\leq \kappa\text{-strat-cl}}$ holds”).*

Proof. By the indestructibility result of [GS], Corollary 6.5 and the fact that the class of $\leq \kappa$ -strategically closed forcing notions is closed under ω iterations using countable support. \square

Remark. The class of $\leq \kappa$ -strategically closed forcing notions includes, for example, all the κ^+ -closed forcing notions.

Separate from the cases just handled is the following result.

Theorem 6.8. *Con(ZFC + “There is a strongly compact cardinal κ ”) \iff Con(ZFC + “There is a strongly compact cardinal κ , indestructible by $\text{Add}(\kappa, 1)$ forcing, such that $\text{MP}_{\text{Add}(\kappa, 1)}$ holds”).*

Proof. By [HAM2], a strongly compact cardinal can be made $\text{Add}(\kappa, 1)$ -indestructible. In this case, $\Gamma = \{\text{Add}(\kappa, 1)\}$, which has just one element. So if anything is forceably necessary by Γ , then forcing once with it will make it happen. So any forcing extension by $\text{Add}(\kappa, 1)$ is a model of $\text{MP}_{\text{Add}(\kappa, 1)}$. \square

6.2 $\text{MP}_{\Gamma(\kappa)}$ with Parameters

Corresponding to the results in Section 6.1 on modifications of the principle MP which pertains to statements without parameters are results analogous to the equiconsistency result regarding $\text{MP}(H(\omega_1))$ —which takes real parameters—in [HAM1]. As before, let κ be a large cardinal. Let $\Gamma(\kappa)$ be a class of forcing notions definable from κ under which the largeness of κ can be made indestructible in some $\Gamma(\kappa)$ -forcing extension. In addition, we now specify a parameter set $S(\kappa)$ to be used in expressing the maximality principle $\text{MP}_{\Gamma(\kappa)}(S(\kappa))$. If we express the largeness of κ with the formula $\sigma(\kappa)$, the generalized theories for which equiconsistency is sought are:

- (1) $\text{ZFC} + \Box_{\Gamma(\kappa)}\sigma(\kappa) + \text{MP}_{\Gamma(\kappa)}(S(\kappa))$
- (2) $\text{ZFC} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”} + \kappa < \delta + \sigma(\kappa)$
- (3) $\text{ZFC} + \text{“}ORD \text{ is Mahlo”} + \sigma(\kappa)$.

Remark. The principle “ ORD is Mahlo” is known as the Lévy Scheme. It says that every club class in ORD contains a regular cardinal.

Theory (1) asserts that the largeness of κ is indestructible under $\Gamma(\kappa)$ -forcing, and that any statement with parameter in $S(\kappa)$ which is $\Gamma(\kappa)$ -forceably necessary is $\Gamma(\kappa)$ -necessary. Theories (2) and (3) are identical to those in [HAM1] with the addition of the largeness of κ . Indeed, the proof that (2) and (3) are equiconsistent follows that in [HAM1]; the forward implication is direct: let $C \subseteq ORD$ be any definable club; $C \cap \delta$ is unbounded in δ so δ , which is regular, is in C . The converse is proved by constructing a model through a compactness argument using Lévy Reflection, which can be done by including κ in an expanded ground model and suitably modifying the argument of Lemma 1.17.

An unsolved problem for the moment is to prove that $Con(1) \implies Con(2)$ for any specific kind of large cardinal. The analogous proof in [HAM1] uses $L[x]$, for real x , as a model of (2) within a model of (1). But here we need to preserve the largeness of κ when we move to an inner model which in fact becomes the core model. Results for strong cardinals have been developed in this model, so it may work here. On the other hand, as of yet, supercompact cardinals have no core model, so for them the problem will probably be more difficult.

At any rate, the best we can do here is prove the direction $Con(2) \implies Con(1)$ for specific cases, which at least gives an upper bound on the consistency strength of $\text{MP}_{\Gamma(\kappa)}(S(\kappa))$, namely, that of a cardinal satisfying property σ together with the Lévy scheme. Rather than prove a general version of this, we will prove it for specific cases, addressing their specific issues. The first result will bound the consistency strength of the principle $\text{MP}_{<\kappa-dc}(H(\kappa^+))$, where κ is a supercompact cardinal and $<\kappa-dc$ is the class of $<\kappa$ -directed closed forcing notions. We will consider the relative consistency of the following theories:

- (1) $\text{ZFC} + \text{“there is a supercompact } \kappa, \text{ indestructible by } <\kappa\text{-directed closed forcing”} + \text{MP}_{<\kappa-dc}(H(\kappa^+))$

- (2) $\text{ZFC} + “\kappa \text{ is supercompact}” + V_\delta \prec V + \kappa < \delta + “\delta \text{ is inaccessible}”$
- (3) $\text{ZFC} + “\kappa \text{ is supercompact}” + “ORD \text{ is Mahlo}”$.

We will prove the implication $\text{Con}(2) \implies \text{Con}(1)$ by constructing a model of theory (1) via a δ iteration, beginning with a model of (2) as ground model. But first we need the following standard lemma to assure us that this iteration does not collapse δ .

Lemma 6.9. *Let δ be inaccessible, $\kappa < \delta$, and $\mathbb{P} = \mathbb{P}_\delta$ a δ -stage, $< \kappa$ -support iterated notion of forcing such that for all $\alpha < \delta$, $\mathbb{P}_\alpha \Vdash “\dot{Q}_\alpha \text{ is in } V_\delta”$ where $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$. Then \mathbb{P} is δ -cc.*

Proof. For all $\alpha < \delta$, \mathbb{P}_α is in V_δ and therefore is δ -cc. This is true by induction: for successor stages each \dot{Q}_α is in V_δ , and at limit stages, \mathbb{P}_α is in V_δ because δ is inaccessible. It remains to show that \mathbb{P}_δ itself is δ -cc. Suppose, towards contradiction, that $A = \{p^\beta \mid \beta < \delta\}$ is an antichain in \mathbb{P}_δ . The iteration is with $< \kappa$ -support. Thus, since $\kappa < \delta$ and δ is inaccessible, $|\delta^{<\kappa}| < \delta$. So by passing to a subset $B \subseteq A$ where $|B| = \delta$, we can assume that $\{\text{supp}(p^\beta) \mid \beta < \delta\}$ forms a Δ -system. Let r be the root of this Δ -system. Since δ is inaccessible, we can fix $\zeta < \delta$ with $r \subset \zeta$.

For any p and p' in B , we have p is incompatible with p' as B is an antichain. However, it is possible to choose such p and p' so that $p \restriction_\zeta$ is compatible with $p' \restriction_\zeta$, since for all $\zeta < \delta$, \mathbb{P}_ζ has the δ -cc. We have $\text{supp}(p) \cap \text{supp}(p') = r \subset \zeta$. But by Lemma 5.11(f) of [KUN], Chapter VIII, this implies that p is incompatible with p' if and only if $p \restriction_\zeta$ is incompatible with $p' \restriction_\zeta$, a contradiction. \square

Theorem 6.10. *If there is a model of $\text{ZFC} + “\kappa \text{ is a supercompact cardinal}” + V_\delta \prec V + “\delta \text{ is inaccessible}” + \kappa < \delta$, then there is a model of $\text{ZFC} + “\kappa \text{ is a supercompact cardinal indestructible by } < \kappa\text{-directed closed forcing}” + \text{MP}_{<\kappa\text{-dc}}(H(\kappa^+))$.*

Proof. Suppose $V \models \text{ZFC} + V_\delta \prec V + “\kappa \text{ is supercompact}” + \kappa < \delta + “\delta \text{ is inaccessible}”$. In V , by the well-known result of Laver [LAV], there is a $< \kappa$ -directed closed notion of forcing $\tilde{\mathbb{P}}$ that produces a model in which the supercompactness of κ is indestructible by further $< \kappa$ -directed closed forcing. Since $\tilde{\mathbb{P}}$ has size $\kappa < \delta$, $\tilde{\mathbb{P}}$ is δ -cc, so it preserves δ and its inaccessibility, together with the statement $\kappa < \delta$.

Let G be V -generic over $\tilde{\mathbb{P}}$. We now work in $V[G]$. By Lemma 1.18, we have that $V_\delta[G] \prec V[G]$. We will construct a $< \kappa$ -support, δ -iteration $\mathbb{P} = \mathbb{P}_\delta$ of $< \kappa$ -directed closed forcing notions as follows. Let Γ be the class of $< \kappa$ -directed closed forcing notions. At stage α let α encode the triple $\langle \beta, \gamma, n \rangle$, where $\beta < \alpha$, $\gamma < \delta$, and $n \in \omega$. Consider $\phi(z)$, where $\phi = \phi_n$, z is the γ^{th} element of $H(\delta)^{V^{\mathbb{P}_\beta}}$ for $\beta < \alpha$. If $\phi(z)$ is Γ -forceably necessary over $V_\delta[G]$, then, in $V_\delta[G]^{\mathbb{P}_\alpha}$, let \dot{Q}_α be the \mathbb{P}_α -name of a forcing notion forcing that $\phi(z)$ is necessary, otherwise let \dot{Q}_α be trivial forcing. This defines $\mathbb{P} = \mathbb{P}_\delta$. Notice that since every \dot{Q}_α is in $V_\delta[G]^{\mathbb{P}_\alpha}$, $|\dot{Q}_\alpha| < \delta$, so \dot{Q}_α is δ -cc. \mathbb{P} is an iteration of these with $< \kappa$ support, so by Lemma 6.9, it is also δ -cc. So δ is not collapsed in the extension, and its cofinality is preserved.

Let H be $V[G]$ generic over \mathbb{P} . Since \mathbb{P} is a δ iteration with $< \kappa$ support of $< \kappa$ -directed closed forcing notions, \mathbb{P} is itself $< \kappa$ -directed closed. And the preceding Laver Preparation $\tilde{\mathbb{P}}$ ensures that κ is still indestructibly supercompact in $V[G][H]$. The final claim is that $V[G][H] \models \text{MP}_\Gamma(H(\delta))$. To show this, suppose $V[G][H]$ satisfies that $\phi(z)$ is Γ -forceably necessary, where z is in $H(\delta)$. Since $|z| < \text{cof}(\delta) = \delta$, z will have been introduced as a \mathbb{P}_β -name at some stage $\beta < \delta$, say, as the γ^{th} element of $V^{\mathbb{P}_\beta}$, and $\phi = \phi_n$ for some n in ω . So there is a stage $\alpha < \delta$ where $\alpha = \langle \beta, \gamma, n \rangle$. $V[G][H_\alpha]$ also satisfies that $\phi(z)$ is Γ -forceably necessary, where H_α is taken to be \mathbb{P}_α generic over $V[G]$ and $V[G][H] = V[G][H_\alpha][H_{TAIL}]$ via the factoring $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{P}}_{TAIL}$. And by elementarity, $V_\delta[G][H_\alpha]$ satisfies that $\phi(z)$ is Γ -forceably necessary. But by the definition of \mathbb{P} , $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$, where $V_\delta[G][H_\alpha] \models \text{"}\dot{Q}_\alpha \text{ forces } \phi(z) \text{ to be } \Gamma\text{-necessary"}$. This implies that by refactoring $\mathbb{P} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}_{TAIL2}$ and taking $H_{\alpha+1}$ to be $\mathbb{P}_{\alpha+1}$ generic over $V[G]$, $V_\delta[G][H_{\alpha+1}] \models \text{"}\phi(z) \text{ is } \Gamma\text{-necessary"}$. Elementarity then gives $V[G][H_{\alpha+1}] \models \text{"}\phi(z) \text{ is } \Gamma\text{-necessary"}$, from which we finally have $V[G][H] \models \text{"}\phi(z) \text{ is } \Gamma\text{-necessary"}$. This gives $V[G][H] \models \text{MP}_\Gamma(H(\delta))$. Notice that \mathbb{P} has $< \kappa$ -directed closed factors that preserve cardinals $< \kappa$ but eventually every ordinal between κ and δ appears as a parameter in $H(\delta)^{V^{\mathbb{P}_\beta}}$ at some stage α where $\beta < \alpha < \delta$. So δ becomes κ^+ in $V[G]$. This finally gives $V[G][H] \models \text{MP}_\Gamma(H(\kappa^+))$ \square

Corollary 6.11. *If there is a model of a supercompact cardinal where the Lévy Scheme holds, then there is a model of a Laver-indestructible supercompact cardinal where $\text{MP}_{<\kappa\text{-dc}}(H(\kappa^+))$ holds.*

Corollary 6.12. *If there is a supercompact cardinal with a Mahlo cardinal above it, then there is a model of a Laver-indestructible supercompact cardinal where $\text{MP}_{<\kappa\text{-dc}}(H(\kappa^+))$ holds.*

Conjecture 1. *The consistency strength of the theory $\text{ZFC} + \text{“there is a supercompact } \kappa, \text{ indestructible by } < \kappa\text{-directed closed forcing”} + \text{MP}_{<\kappa\text{-dc}}(H(\kappa^+))$ is the same as that of the theory $\text{ZFC} + \text{“}\kappa \text{ is supercompact”} + V_\delta \prec V + \kappa < \delta + \text{“}\delta \text{ is inaccessible”}$.*

Looking next at the Gitik-Shelah result of [GS] for strong cardinals, we see that the previous argument adapts to handle this case. Again, we will consider the class $\Gamma_{\leq \kappa\text{-strat-cl}}$ of $\leq \kappa$ -strategically closed forcing notions, a strengthening of the Prikry condition Gitik and Shelah impose. An important difference between this class and the class of $< \kappa$ -directed closed forcing notions is that the most general parameter set of $\text{MP}_{\leq \kappa\text{-strat-cl}}(X)$ is $H(\kappa^{++})$, the sets of hereditary size less than κ^{++} , by the usual consideration: one can regard a parameter from this set as encoded by a subset of κ^+ , any larger parameter set will have members of size greater than κ^+ . Suppose, toward contradiction, that $|x| > \kappa^+$ for some parameter x in a parameter set X . Such a parameter x can have its cardinality collapsed by $\leq \kappa$ -strategically closed forcing. So the statement $|x| \leq \kappa^+$ is $\Gamma_{\leq \kappa\text{-strat-cl}}$ -forceably necessary. By applying $\text{MP}_{\leq \kappa\text{-strat-cl}}(X)$, this falsifies the statement that $|x| > \kappa^+$.

Theorem 6.13. *If there is a model of $\text{ZFC} + \text{“}\kappa \text{ is a strong cardinal”} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”} + \kappa < \delta$, then there is a model of $\text{ZFC} + \text{“}\kappa \text{ is a strong cardinal indestructible by } \leq \kappa\text{-strategically closed forcing”} + \text{MP}_{\leq \kappa\text{-strat-cl}}(H(\kappa^{++}))$.*

Proof. Here we emulate the previous proof, performing the necessary forcing preparation $\tilde{\mathbb{P}}$ in a model of theory (2) ($\text{ZFC} + \text{“}\kappa \text{ is a strong cardinal”} + V_\delta \prec V + \text{“}\delta \text{ is inaccessible”} + \kappa < \delta$). By elementarity, the construction can be assumed to take place in V_δ . This means $\tilde{\mathbb{P}}$ has cardinality below δ , and is therefore δ -cc and the resulting forcing extension $V[G]$, where G is M -generic over $\tilde{\mathbb{P}}$, preserves theory (2). Of course, indestructibility of the strongness of κ also holds in the extension.

Let G be V -generic over $\tilde{\mathbb{P}}$. We now work in $V[G]$. By Lemma 1.18, we have that $V_\delta[G] \prec V[G]$. We will construct a $\leq \kappa$ -support, δ -iteration $\mathbb{P} = \mathbb{P}_\delta$ of $\leq \kappa$ -strategically closed forcing notions as follows. At stage α let α encode the triple $\langle \beta, \gamma, n \rangle$, where $\beta < \alpha$, $\gamma < \delta$, and $n \in \omega$. Consider $\phi(z)$, where $\phi = \phi_n$, z is the γ^{th} element of $H(\delta)^{V^{\mathbb{P}_\beta}}$ for $\beta < \alpha$. If $\phi(z)$ is $\Gamma_{\leq \kappa\text{-strat-cl}}$ -forceably necessary over $V_\delta[G]$, then, in $V_\delta[G]^{\mathbb{P}_\alpha}$, let $\dot{\mathbb{Q}}_\alpha$ be the \mathbb{P}_α -name of a $\leq \kappa$ -strategically closed forcing notion forcing that $\phi(z)$ is necessary, otherwise let $\dot{\mathbb{Q}}_\alpha$ be trivial forcing. This defines $\mathbb{P} = \mathbb{P}_\delta$. Notice

that since every \dot{Q}_α is in $V_\delta[G]^{\mathbb{P}_\alpha}$, $|\dot{Q}_\alpha| < \delta$, so \dot{Q}_α is δ -cc. \mathbb{P} is an iteration of these with $\leq \kappa$ support, so by Lemma 6.9, using κ^+ in place of κ , it is also δ -cc. So δ is not collapsed in the extension, and its cofinality is preserved.

Let H be $V[G]$ generic over \mathbb{P} . An argument similar to that of the previous theorem shows that $V[G][H] \models \text{MP}_{\leq \kappa\text{-strat-cl}}(H(\kappa^{++}))$. \square

Remark. Notice that \mathbb{P} , in this proof, contains $\leq \kappa$ -strategically closed forcing factors that preserve cardinals $< \kappa^+$ but eventually collapse all cardinals between κ^+ and δ . So δ becomes κ^{++} in $V[G]$.

Corollary 6.14. *If there is a model of a strong cardinal where the Lévy Scheme holds, then there is a model of a Gitik-Shelah-indestructible strong cardinal κ where $\text{MP}_{\leq \kappa\text{-strat-cl}}(H(\kappa^{++}))$ holds.*

Corollary 6.15. *If there is a strong cardinal with Mahlo cardinal above it, then there is a model of a Gitik-Shelah-indestructible strong cardinal κ where $\text{MP}_{\leq \kappa\text{-strat-cl}}(H(\kappa^{++}))$ holds.*

Conjecture 2. *The theory $\text{ZFC} + \text{“}\kappa \text{ is a strong cardinal, indestructible by } \leq \kappa\text{-strategically closed forcing”} + \text{MP}_{\leq \kappa\text{-strat-cl}}(H(\kappa^{++}))$ is the same consistency strength as the theory $\text{ZFC} + \text{“}\kappa \text{ is strong”} + V_\delta \prec V + \kappa < \delta + \text{“}\delta \text{ is inaccessible”}$.*

Other results could follow in this vein. Hamkins’ own results on establishing indestructibility of large cardinals by the Lottery preparation in [HAM2] should be usable here. By performing all such preparations in V_δ , they have the δ -cc property. All that is required is closure under appropriate support δ -iterations of the class of forcing notions for which indestructibility is asserted. The resulting forcing notion will be δ -cc, preserving δ , and a generic extension over this forcing notion will satisfy the corresponding maximality principle.

Bibliography

- [ASP] D. Aspero, *Bounded Forcing Axioms and the Continuum*, Doctoral Dissertation, Universitat de Barcelona, May, 2000.
- [BELL] J. L. Bell, *Boolean-Valued Models and Independence Proofs in Set Theory*, Oxford University Press, New York, 1985.
- [FM] M. Fitting and R. Mendelsohn, *First-Order Modal Logic*, Kluwer, Dordrecht, 1999.
- [GS] M. Gitik, S. Shelah, *On certain indestructibility of strong cardinals and a question of Hajnal*, Arch. Math. Logic 28 (1989): 35-42.
- [HAM1] J. Hamkins, *A simple maximality principle*, Journal of Symbolic Logic 68(2) (2003): 527-550.
- [HAM2] J. Hamkins, *The lottery preparation*, Annals of Pure and Applied Logic 101 (2000): 103-146.
- [JECH] Thomas Jech, *Set Theory, 3d Edition*, Springer, New York, 2003.
- [KAN] Akihiro Kanamori, *The Higher Infinite*, Springer, New York, 1997.
- [KUN] Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.
- [LAV] R. Laver, *Making the supercompactness of κ indestructible under κ -directed closed forcing*, Israel J. Math. 29 (1978): 385-388.
- [SV] J. Stavi and J. Väänänen, *Reflection principles for the continuum*, Logic and Algebra 302 (2002): 59-84.

- [TEN] Stanley Tennenbaum, *Souslin's Problem*, Proc. Nat. Acad. U.S.A. 59 (1968): 60-63.
- [TF] Stevo Todorcevic and Ilijas Farah, *Some Applications of the Method of Forcing*, YENISEI, Moscow, 1995.
- [TOD] Stevo Todorcevic, *Coherent Sequences*, preprint.